Resource Selection Games with Unknown Number of Players

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Abstract

In the context of pre-Bayesian games we analyze resource selection systems with unknown number of players. We prove the existence and uniqueness of a symmetric safety-level equilibrium in such games and show that in a linear model every player benefits from the common ignorance about the number of players. In order to analyze such games we generalize the theory of equilibrium in general pre-Bayesian games.

1. Introduction

There has been much interest in the recent years in work bridging computer science and game theory. Most of this work is concerned with the analysis of multi-agent systems, where the agents are selfish, and each agent attempts to maximize his own utility or minimize his own cost. Hence, it is natural that work in multi-agent systems adopted the notion of equilibrium as the type of solution concept to be discussed. On the other hand, when dealing with settings of incomplete information, work in computer science typically adopts non-Bayesian models, where qualitative decision making rules (without probabilistic assumptions about the environment) are used. However, in a multi-agent setup with incomplete information the non-Bayesian approach might severely restrict the type of analysis that may be carried out. Indeed, most previous work on games with incomplete information in computer science adopted dominant strategy equilibrium or ex-post equilibrium as the major solution concepts. This paper is part of a line of research that introduces non-Bayesian equilibrium concepts as appropriate solutions for dealing with incomplete information in multi-agent systems. One interesting property is that (as we show) the related (non-Bayesian) equilibrium always exist.[1] The main challenge remained is to consider concrete powerful settings, where the related (non-Bayesian) equilibrium concepts can be discussed, and lead to illuminating results. In this paper we deal with such setting, resource selection games, which is central to the computer science literature. We present an analysis of resource selection games with incomplete information, using the non-Bayesian approach, yielding several highly powerful and surprising results.

In a *resource selection system*, Γ, there is a set of m resources, $j \in \{1, \cdots, m\}$. Each resource j is associated with a cost function $w_j : \{1, 2, \cdots, \cdots\} \to \Re$, where $w_j(k)$ is the cost for every user of resource j if there are k users. Together with a set of n players a resource selection system defines a game in strategic form– *a resource selection game*, $\Gamma(n)$. The action set[2] of every player i in $\Gamma(n)$ is the set of resources M, and the cost of i depends, via the resource-cost functions on the resource she chooses and on the number of other players who choose this resource. Thus, resource selection games are special type of congestion games [22, 17]. A resource selection game is also referred to as a simple congestion game.[3]. In many situations the assumption that every player knows the number of players is not reasonable. The goal of the current paper is to analyze resource selection games with unknown number of players. One approach for analyzing such situations is the Bayesian approach, where it is assumed that the distribution of the random set of players is commonly known. In this approach one is interested in Bayesian equilibrim. [4] The goal

1 In a finite setup.

2 When dealing with games in strategic form the choice set of a player is referred to as an action set or as a strategy set. We use "actions" rather than "strategies" because we keep the notion of strategy to describe the choice set of a player in "bigger" games.

3 Simple congestion games and their generalization to player-specific games were discussed in [21, 15, 8, 26, 10]. Simple congestion games were also discussed in the price of anarchy literature, e.g., [11, 2]

4 See [14, 13, 6] for such analysis in the context of auctions, and [18,

of this paper is to analyze resource selection games in the non-Bayesian setup in which the players do not have probabilistic information about the other players in the game. We will use the model of pre-Bayesian games.

We remark that games can be analyzed with either payoff functions or cost functions. Most of the general theory has been developed with payoff functions. However, congestion games have been mainly analyzed using cost functions. Translating results proved in one of the setups to the other setup is obvious in most cases.[5] We follow the tradition of previous literature in the sense that the general theory is discussed with payoff functions, while resource selection games are discussed with cost functions.

In a classical pre-Bayesian game there is a fixed set of players $N = \{1, 2, \cdots, n\}$, each of them is endowed with a set of actions, $x_i \in X_i$. There is a set of states $\omega \in \Omega$. The payoff of player i, $u_i(\omega, x_1, x_2, \cdots, x_n)$ depends on the realized state, on her choice of action, x_i, and on the choices of all other players. However, the realized state is not known to the players. Every player receives a state-correlated signal, $t_i = \tilde{t}_i(\omega)$ on which she conditions her action. A pre-Bayesian game becomes a Bayesian game when a commonly known probability measure on the set of states is added to the system. In pre-Bayesian games one can deal with ex-post equilibrium. It is well-known that an ex-post equilibrium in a pre-Bayesian game is a Bayesian equilibrium for every choice of prior probability and vice versa. Indeed, the classical literature in economics/game theory actually discussed ex-post equilibrium in a particular Bayesian game, and defined it as a Bayesian equilibrium which is robust to changes in the prior probability. Only recently the concept of pre-Bayesian games have been explicitly defined.[6] Unfortunately, ex-post equilibrium does not exist even in very simple pre-Bayesian games. A strategy of player i in a pre-Bayesian or Bayesian game is a function b_i that assigns an action (or a mixed action), $x_i = b_i(t_i)$ to every possible signal t_i. In all concepts of equilibrium it is assumed that a player knows the other players' strategies. Hence, when a player receives a signal, t_i, she is facing a classical decision problem with uncertainty about the true state. She knows, however, that the true state should be compatible with her signal. That is, the true state is one of the states in $\Omega(t_i) \subseteq \Omega$ that yields the signal, and she is optimizing her choice in this decision problem. In a Bayesian equilibrium, optimizing means maximizing expected utility over $\Omega(t_i)$ with respect to the given prior probability. In an ex-post equilibrium, optimizing means maximizing utility for every compatible state.

Another equilibrium concept for pre-Bayesian games was recently defined in [4] - a minimax-regret equilibrium. In such an equilibrium optimizing means choosing an action that minimizes the maximal regret over compatible states. It was proved in [4] that a minimax-regret equilibrium always exist in a finite setup[7] when players are using mixed strategies.[8] An equilibrium concept for forms of pre-Bayesian games has been already defined in [25, 24, 23] in the context of work on artificial social systems, and [12] defined another equilibrium concept for particular pre-Bayesian auctions. Both groups of authors used distinct generalizations of maximin equilibrium, which was recently defined for classical pre-Bayesian games in [1]. In a maximin equilibrium optimizing means maximizing utility in the worst case scenario. The min in a maximin equilibrium ranges over all states that are compatible with the signal t_i, and the max ranges over all actions, X_i. Therefore, in a cost model the maximin equilibrium is actually a minimax equilibrium. We therefore prefer to use the term *safety level equilibrium* for both payoff and cost models. In [1] the authors mainly deal with the non-private type case, in which all players have the same signaling function. However, they prove existence results for the classical pre-Bayesian games described above.

The existence of multiple equilibria has always been a problematic issue in game theory. For pre-Bayesian games we also face the problem of multiple types of equilibrium. The question of which type of equilibrium in pre-Bayesian games makes a better prediction is yet to be explored. It is observed however, by [4] and [1] that every ex-post equilibrium is both a minimax-regret equilibrium and a safety level equilibrium.

In order to analyze resource selection games with unknown number of players we mainly use the safety level equilibrium solution. However, the model of pre-Bayesian games described above is not applicable for that matter, since the set of players in this model is fixed. We therefore generalize the model of pre-Bayesian games. In our generalized model, the set of players and the action sets of the players are state dependent.

We prove existence results for maximin and minimax-regret equilibrium in the generalized model.[9]. All proofs of existence (in [4], [1] and ours) make use of (variations of) Kakutani fixed point theorem. For the sake of completeness we also present and prove an existence theorem for a third, and new equilibrium concept for pre-Bayesian games

19, 20] for such analysis in the context of elections.

5 An exception is the price of anarchy theory.

6 Pre-Bayesian games have been also called *games in informational form* and *games without probabilistic information* [9, 7], *games with incomplete information with strict type uncertainty* [4], and *distribution-free games with incomplete information* [1].

7 Namely, the number of players and the number of states are finite.

8 Actually, the proof in [4] is given for a less general model of pre-Bayesian games than the one described above, but in this paper we extend their proof to the more general case.

9 We also allow compact convex sets of actions.

- the competitive ratio equilibrium. The competitive ratio approach is relatively common in computer science.[10]

Once we have the right tools we proceed to analyze resource selection games with unknown number of players. We focus on a model with increasing resource cost functions. In order to derive results for the case in which the number of players is unknown we prove several results, some of them are interesting for themselves, about classical resource selection games with known number of players. In particular we prove that every resource selection game possesses a unique mixed-action symmetric equilibrium. When the number of players is k, the unique symmetric equilibrium mixed-action of every player is denoted by p^k. That is, $p^k = (p_1^k, p_2^k, \cdots, p_m^k)$, where p_j^k is the probability that a player chooses resource j.

In our pre-Bayesian model all active players know a common bound, n, on the number of active players, but the players do not know the true number of players, say k, $k \leq n$. Hence, the only signal a player receives is an "activity" signal. A state in this pre-Bayesian game is the set of active players. We prove that a resource selection game with unknown number of players has a unique symmetric safety level equilibrium. In this equilibrium, every active player is using the unique symmetric-equilibrium mixed-action, p^n in the game in which the number of players is commonly known and equals n.

Hence, when the number of players is k, every player uses p^k when the number of players is commonly known, and every player is using p^n when the number of players is unknown. Surprisingly, the lack of knowledge makes each of the players better off in the linear system, in which the resource cost functions are linear! That is, we show that in the linear model, when there are k players, and each of them is using p^n, the cost of each player is at most the cost he obtains in the unique symmetric equilibrium, p^k. Under very modest assumptions every player is strictly better off.

The above results are applicable to a mechanism design setup in which the organizer knows the number of active players, and the players do not know this number. If the goal of the organizer is to maximize revenue then he is better off revealing his private information.[11] If his goal is to maximize social surplus then he should not reveal the information. In order to estimate the gain of the players resulting from their ignorance we investigate in the last section the function $c^k(p^n)$ for $n \geq k$, where $c^k(p^n)$ is the cost for a player in a k-player setup when all players play the mixed-action associated with the symmetric equilibrium of the corresponding n-player setup.

In the full paper we also analyze the minmax-regret equilibrium of the related setting. We show that the phenomenon described above does not hold for this solution concept. The precise relationships between the linear cost functions of the different resources determine whether knowing the number of participants is helpful or harmful for the players. Many of the proofs are omitted in this conference version of the paper.

10 See e.g., [3].

11 This is indirectly related to the Linkage Principle, in auction theory [16].

2. Background

A game in strategic form is a tuple, $\Gamma = (N, (X_i)_{i\in N}, (u_i)_{i\in N})$. N is a nonempty set of players, X_i is a nonempty set of actions for player i, and $u_i : X \to \Re$ is the payoff function of i, where $X = \times_{i\in N} X_i$ is the set of action profiles. Hence $u_i(x)$ is the payoff of player i when the profile of actions $x \in X$ is played. Γ is a *finite* game if N and X are finite sets.

Let $x \in X$ denote a profile of actions. For each $i \in N$ we let $x_{-i} = (x_j)_{j\in N/\{i\}}$ denote the actions played by everyone but i. Thus $x = (x_i, x_{-i})$. An action profile $x \in X$ is in *equilibrium* if $u_i(x_i, x_{-i}) \geq u_i(y_i, x_{-i})$ for every player $i \in N$ and for every $y_i \in X_i$.

A *permutation* of the set of players is a one-to-one function from N onto N. For every permutation π and for every action profile $x \in X$ we denote by πx the permutation of x by π. That is, $(\pi x)_i = x_{\pi(i)}$ for every player i. Γ is a *symmetric game* if for every player i, for every action profile x, and for every permutation π

$$u_i(\pi x) = u_{\pi(i)}(x).$$

A *symmetric action profile* is an action profile x such that $x_i = x_s$ for every $i, s \in N$. x is *a symmetric equilibrium* if it is both an equilibrium profile and a symmetric action profile .

For any finite set C, $\Delta(C)$ denotes the set of probability distributions over C. Let $\Gamma = (N, (X_i)_{i\in N}, (u_i)_{i\in N})$ be a finite game in strategic form. Every $p_i \in \Delta(X_i)$ is called a *mixed action* for i. $p_i(x_i)$ is the probability that player i plays action x_i. Every vector $p \in \Delta = \times_{i\in N}\Delta(X_i)$ defines a probability distribution over X; The probability, $p(x)$ of $x \in X$ is $\prod_{i\in N} p_i(x_i)$.

Let u_i^m be the expected payoff function defined on Δ by u_i. That is, $u_i^m(p) = E_p(u_i)$.

The game $\Gamma^m = (N, (\Delta(X_i))_{i\in N}, (u_i^m)_{i\in N})$ is called the *mixed extension* of Γ. A mixed action for player i in the game Γ is an action for player i in the game Γ^m.

A *mixed action equilibrium* in a finite game Γ is defined to be an equilibrium in Γ^m. It is well-known (see e.g, [5]) that every finite symmetric game in strategic form possesses a symmetric mixed action equilibrium.

3. Resource Selection Games with Known Number of Players

In a *resource selection system*, $\Gamma = (m, (w_j)_{j=1}^m)$ there is a set of resources, $M = \{1, \cdots, m\}$, $m \geq 1$. Every resource j is associated with a cost function $w_j : \{1, 2, \cdots\} \to \Re$. $w_j(k)$ is the cost for every user of resource j if there are k users. Together with the set of players $N_n = \{1, \cdots, n\}$ a resource selection system defines a game in strategic form– *a resource selection game* $\Gamma(n)$. The action set of every player i in $\Gamma(n)$ is the set of resources M, and the cost of i depends on the resource she chooses and on the number of other players that choose this resource via the resource-cost functions. That is, $X_i^n = M$ for every $1 \leq i \leq n$, and $c_i^n(x) = w_{x_i}(\sigma_{x_i}(x))$, where for every resource j and for every action profile $x \in \times_{i=1}^n X_i^n = M^n$, $\sigma_j(x)$ is the number of all players s for which $x_s = j$. Obviously every resource selection game is a finite symmetric game.

Let $p \in \Delta(M)$ be a mixed action of an arbitrary player. That is, $p = (p_1, \cdots, p_m)$, where p_j is the probability that a player who uses the mixed action p will select resource j. We denote the support of p by $supp(p)$. That is $supp(p) = \{j \in M | p_j > 0\}$. Denote by $c^n(p, j)$ the expected cost of a player that chooses resource j when each of the other $n-1$ players in $\Gamma(n)$ is using p. Let $c^n(p)$ be the expected cost of every player when each of the n players in $\Gamma(n)$ is choosing p.

For every $n \geq 1$, and for every $0 \leq \alpha \leq 1$. Let $Y_\alpha^n \sim Bin(n, \alpha)$ be a binomial random variable. That is, $f_\alpha^n(k) = P(Y_\alpha^n = k) = \binom{n}{k}\alpha^k(1-\alpha)^{n-k}$ for every $0 \leq k \leq n$. Let $F_\alpha^n(k) = P(Y_\alpha^n \leq k)$ be the distribution function of Y_α^n. Obviously

$$c^n(p, j) = E(w_j(1 + Y_{p_j}^{n-1})), \tag{1}$$

where E stands for the expectation operator. That is,

$$c^n(p, j) = \sum_{s=0}^{n-1} w_j(s+1) f_{p_j}^{n-1}(s). \tag{2}$$

Let $(q, \cdots, q) \in \Delta(M)^n$ be a symmetric mixed-action equilibrium profile in $\Gamma(n)$. We will refer to q as a *symmetric-equilibrium action*.

Theorem 1 *Every resource selection game with at least two players ($n \geq 2$) with increasing* [12] *resource cost functions possesses a unique symmetric mixed-action equilibrium.*

In order to prove Theorem 1 we need some preparations. We prove two lemmas, and a corollary to these lemmas. The proofs of Theorem 1, of the lemmas, and of the corollary are given in the full version.

12 That is, $w_j(k) < w_j(k+1)$ for all j and k.

Lemma 1 *Let $n \geq 1$. $F_\alpha^n(k)$ is a strictly decreasing function of α for every $0 \leq k \leq n-1$.*

Lemma 2 *Let $\Gamma(n)$, $n \geq 2$ be a resource selection game. Let $q, p \in \Delta(M)$ be mixed actions, and let Let $j \in M$ be a resource such that w_j is increasing in $\{1, 2, \cdots, n\}$. If $p_j > q_j$ then $c^n(p, j) > c^n(q, j)$.*

Corollary 1 *Let $\Gamma(n)$, $n \geq 2$ be a resource selection game with increasing resource cost functions. All symmetric equilibrium actions in $\Gamma(n)$ have the same support.*

For every $n \geq 1$ we will denote the unique symmetric equilibrium in $\Gamma(n)$ by p^n, and we denote by $c^n = c^n(p^n)$ the equilibrium cost of a player in $\Gamma(n)$.

We say that a resource cost function w_j is *convex* if it can be extended to a convex function on $[1, \infty)$.

The following lemma will be useful later.

Lemma 3 *Let $\Gamma = (m, (w_j)_{j=1}^m)$ be a resource selection system, with increasing and convex cost functions. There exists an integer $K \geq 2$, $K = K(\Gamma)$ such that for every $n \geq K$, the unique symmetric-equilibrium action in the game $\Gamma(n)$, $p^n \in \Delta(M)$ has a full support. That is, $p_r^n > 0$ for every $1 \leq r \leq m$.*

Proof: Recall that p^n is the unique symmetric-equilibrium action in $\Gamma(n)$, and that $c^n = c^n(p^n)$ is the symmetric-equilibrium cost of every player. As p^n is in equilibrium, $c^n(p^n, j) = c^n$ for every $j \in supp(p^n)$. For every resource j we denote by w_j the convex extension of w_j to $[0, \infty)$. As w_j is convex,

$$c^n(p^n, j) = E(w_j(1 + Y_{p_j^n}^{n-1})) \geq w_j(1 + E(Y_{p_j^n}^{n-1})) =$$
$$w_j(1 + p_j^n(n-1)),$$

where the first equality follows from (1), the inequality follows from the convexity of w_j, and the last equality follows from the well-known fact that

$$E(Y_\alpha^n) = \alpha n. \tag{3}$$

Obviously, there exists $j \in supp(p)$ for which $p_j^n \geq \frac{1}{m}$. For this resource j

$$c^n = c^n(p^n, j) \geq w_j(1 + \frac{1}{m}(n-1)) \geq$$
$$\min_{r=1}^{m} w_r(1 + \frac{1}{m}(n-1)).$$

Since w_j is increasing and convex, $\lim_{n\to\infty} w_j(n) = \infty$ for every resource j. Therefore $\lim_{n\to\infty} c^n = \infty$. Hence, there exists K such that for every $n \geq K$ $c^n > \max_{j=1}^m w_j(1)$. We claim that for every $n \geq K$, $p_r^n > 0$ for every $1 \leq r \leq m$. Indeed, if $p_r^n = 0$ for some r then because $c^n > w_r(1)$, a player will decreases her cost by deviating from p^n to r (assuming every other player is using p^n). This contradicts p^n being a symmetric-equilibrium action. ∎

4. Equilibrium in Pre-Bayesian Games

In this section we define a general model of pre-Bayesian games, and we present and prove existence results for minimax-regret, safety-level, and competitive-ratio equilibria in such games.

A *pre-Bayesian game* is a tuple $G = (N, \Omega, (X_i)_{i\in N}, (S_i)_{i\in N}, (T_i)_{i\in N} (\tilde{t}_i)_{i\in N}, (u_i)_{i\in N}, (e)_{i\in N})$, where

- N is the set of potential players.
- Ω is the set of *states*.
- X_i is the set of possible actions of agent $i --$ Only a subset of X_i will be available for i at a given state ω.
- The symbol e_i stands for a dummy action for i. It does not belong to X_i. Let $Z_i = X_i \cup \{e_i\}$,and let $Z = \times_{i\in N} Z_i$.
- $T_i \subseteq 2^{X_i} \times S_i$ is the set of *types* of i. Hence, every type of i is a pair $t_i = (t_i^1, t_i^2)$, where $t_i^1 \subseteqq X_i$ and $t_i^2 \in S_i$. t_i is an *active* type if $t_i^1 \neq \emptyset$.
- $\tilde{t}_i : \Omega \to T_i$ is the *signaling function* of i; $\tilde{t}_i(\omega) = (t_i^1(\omega), t_i^2(\omega)) = (X_i(\omega), s_i(\omega))$. For every t_i we denote by $\Omega(t_i)$ the set of states that can generate t_i. That is $\Omega(t_i) = \{\omega | \tilde{t}_i(\omega) = t_i\}$. We assume that $\Omega(t_i) \neq \emptyset$ for every $t_i \in T_i$.

  Player i is *active* at ω if $\tilde{t}_i(\omega)$ is an active type. The set of all active players at ω is denoted by $N(\omega)$.
- $u_i : \Omega \times Z \to \Re$ is the *payoff function* of i.

The pre-Bayesian game proceeds as follows. Nature chooses $\omega \in \Omega$. Every player $i \in N(\omega)$ chooses an action $x_i \in X_i(\omega)$. Every non-active player j does nothing. It is modelled by choosing e_j. Hence, the players form a profile $z \in Z$. Every active player i receives $u_i(\omega, z)$. A *strategy* of i is a function $b_i : T_i \to Z_i$ such that $b_i(\emptyset, s_i) = e_i$ for every $s_i \in S_i$, and $b_i(t_i) \in t_i^1$ whenever $t_i^1 \neq \emptyset$. We denote by Σ_i the set of strategies of player i ,and let $\Sigma = \times_{i\in N}\Sigma_i$.

Let G be a pre-Bayesian game. We say that G is a *finite* if the set of potential players, the set of states , and the strategy sets are finite. We say that G is a *compact-continuous pre-Bayesian game* if the set of players and the set of states are finite, X_i is a compact subset of some Euclidean space[13], for every type t_i, t_i^1 is a compact subset of X_i, and for every $\omega \in \Omega$, the restriction of $u_i(\omega, *)$ to $X_i \times Z_{-i}$ is continuous in x_i.

A compact continuous pre-Bayesian game is *concave* if X_i is a convex subset of some Euclidean space, every $t_i^1 \subseteq X_i$ is convex , and for every $\omega \in \Omega$, the restriction of $u_i(\omega, *)$ to $X_i \times Z_{-i}$ is concave in x_i.

Let $G = (N, \Omega, (X_i)_{i\in N}, (S_i)_{i\in N}, (T_i)_{i\in N}, (\tilde{t}_i)_{i\in N}, (u_i)_{i\in N}$,e) be a finite pre-Bayesian game. Denote $\Delta_i = \Delta(X_i)$, and for every $Y_i \subseteq X_i$ let $O_i(Y_i)$ be the set of all probability distributions in Δ_i that vanish outside Y_i. Note that $O_i(\emptyset) = \emptyset$. Let $T_i^m = \{(O_i(t_i^1), t_i^2) | (t_i^1, t_i^2) \in T_i\}$. Define $\tilde{t}_i^m : \Omega \to T_i^m$ as follows: $\tilde{t}_i^m(\omega) = (O_i(\tilde{t}_i^1(\omega)), \tilde{t}_i^2(\omega))$. The pre-Bayesian game $G^m = (N, \Omega, (\Delta_i)_{i\in N}, (S_i)_{i\in N}, (T_i)_{i\in N}, (\tilde{t}_i^m)_{i\in N}, (u_i^m)_{i\in N}$,e) is called the *mixed extension* of G, where u_i^m is the expected-payoff function defined by u_i. Obviously G^m is a concave pre-Bayesian game. Every strategy of i in G^m is called a *mixed-strategy for i in G*.

13 Or of some linear topological space.

4.1. Safety-level equilibrium

Safety-level equilibrium were defined by Aghassi and Bertsimas in ([1]). We use their definition for our general model of pre-Bayesian games.

Let G be a compact-continuous pre-Bayesian game. Let $b_{-i} = (b_j)_{j\in N\setminus i}$ be a profile of strategies of all players but i, and let $t_i = (Y_i, s_i)$ be an active type for i. For every $y_i \in Y_i$ the worst case payoff of i is

$$V_i(t_i, b_{-i}, y_i) = \min_{\omega\in\Omega(t_i)} u_i(w, y_i, b_{-i}(\tilde{t}_{-i}(\omega)).$$

Obviously V_i is continuous on Y_i. We say that $y_i^* \in Y_i$ is *optimal* for an active type $t_i = (Y_i, s_i)$ given b_{-i} if the maximal value of $V_i(t_i, b_{-i}, y_i)$ over $y_i \in Y_i$ is attained at y_i^*. A strategy of player i, b_i, is a safety-level best-response to b_{-i} if for every active type t_i, $b_i(t_i)$ is optimal for t_i given b_{-i}. A strategy profile $b = (b_i)_{i\in N}$ is called a *safety-level equilibrium* if for every i, b_i is a safety-level best-response to b_{-i}.

Hence, b is a safety-level equilibrium if and only if for every ω, and for every player i, which is active at ω, $b_i(\tilde{t}_i(\omega))$ is optimal for $\tilde{t}_i(\omega)$ given b_{-i}.

A safety-level equilibrium in a pre-Bayesian game with exactly one state is simply a Nash equilibrium in this game. We next show that safety-level equilibria exist in every concave game.

Theorem 2 *Every concave pre-Bayesian game possesses a safety-level equilibrium.*

Proof: For every i, let T_i^a be the set of active types, and let $T_i^d = T_i \setminus T_i^a$ be the set of dummy types. Every $b_i \in \Sigma_i$ is determined by its restriction to T_i^a. Note that Σ_i is a closed convex subset of $X_i^{T_i^a}$, which is a convex and compact set because it is a cartesian products of such sets. In this sense, every Σ_i and Σ are convex and compact spaces. For every $b \in \Sigma$ and for every $i \in N$ let $B_i(b) \subseteq \Sigma_i$ be the set of all $d_i \in \Sigma_i$, which are best response to b_{-i}. Let $B(b) = \times_{i\in N} B_i(b) \subseteq \Sigma$. It is standard to check that the correspondence $b \to B(b)$ satisfies the conditions of Kakutani's fixed point theorem. That is, it is upper hemicontinuous, and $B(b)$ is a nonempty compact convex subset of Σ

for every $b \in \Sigma$. Therefore there exists a fixed point b, that is, $b_i \in B_i(b)$ for every $i \in N$. Obviously such a fixed point is a safety-level equilibrium. ▮

Hence, if G is a finite game, G^m possesses a safety-level equilibrium. Every such an equilibrium is called a *mixed-strategy safety-level equilibrium* in G.

4.2. Minimax-regret equilibrium

Minimax-regret equilibrium were defined by Hyafil and Boutilier in [4]. We use their definition for our general model of pre-Bayesian games.

Let G be a compact-continuous pre-Bayesian game. Let $b_{-i} = (b_j)_{j \in N \setminus i}$ be a profile of strategies of all players but i, let t_i be an active type of i, and let $\omega \in \Omega(t_i)$. The *regret* of $x_i \in t_i^1$ at w is defined as:

$$R(x_i, w, t_i, b_{-i}) =$$

$$\max_{z_i \in t_i^1} [u_i(w, z_i, b_{-i}(\tilde{t}_{-i}(\omega)) - u_i(w, x_i, b_{-i}(\tilde{t}_{-i}(\omega)))].$$

The *maximal regret* of $x_i \in t_i^1$ over all $\omega \in \Omega(t_i)$ is denoted by $MR(x_i, t_i, b_{-i})$. That is:

$$MR(x_i, t_i, b_{-i}) = \max_{\omega \in \Omega(t_i)} R(x_i, w, t_i, b_{-i}).$$

We say that $y_i \in t_i^1$ is a minimax regret strategy at t_i given b_{-i} if the minimal value of $MR(x_i, t_i, b_{-i})$ over $x_i \in t_i^1$ is attained at y_i. A strategy b_i is a minimax regret best response to b_{-i} if for every active type t_i, $b_i(t_i)$ is a minimax regret strategy at t_i given b_{-i}. b is a *minimax regret equilibrium* if for every player i b_i is a minimax regret best response to b_{-i}. The next theorem shows that minimax-regret equilibria exist in every concave pre-Bayesian game. The proof is similar to the proof of theorem 2.

Theorem 3 *Every concave pre-Bayesian game possesses a minimax-regret equilibrium.*

4.3. Competitive ratio equilibrium

Competitive ratio equilibrium resembles the minimax-regret equilibrium. They differ only in the definition of regret.

Let G be a compact-continuous pre-Bayesian game, where all payoff functions are positive. Let $b_{-i} = (b_j)_{j \in N \setminus i}$ be a profile of strategies of all players but i, let t_i be an active type of i, and let $\omega \in \Omega(t_i)$. Replace the definition of $R(x_i, w, t_i, b_{-i})$ in the previous section with the following:

$$\hat{R}(x_i, w, t_i, b_{-i}) = \max_{z_i \in t_i^1} \frac{u_i(w, z_i, b_{-i}(\tilde{t}_{-i}(\omega))}{u_i(w, x_i, b_{-i}(\tilde{t}_{-i}(\omega)))}.$$

Note that $\hat{R}$ is well defined since all payoff functions are positive.A strategy profile b is a competitive-ratio equilibrium if for every player i b_i is a minimax-regret best response to b_{-i} with respect to the regret function $\hat{R}$.

Theorem 4 *Every concave game in pre-Bayesian game with positive payoffs functions possesses a competitive ratio equilibrium.*

The proof follows from Theorem 3 by applying the logarithmic function to the payoff functions.

5. Resource Selection Games with Unknown Number of Players

Consider a fixed resource selection system, Γ with the set of resources $M = \{1, \cdots, m\}$, $m \geq 1$, and resource cost functions $(w_j)_{j=1}^m$.

We proceed to describe our model of resource selection games with unknown number of players. Let $N = \{1, 2, \cdots, n\}$, $n \geq 1$. be the set of potential players. The set of states, Ω is the set $2^N \setminus \{\emptyset\}$ of all nonempty subsets of N. The set of actions of player i is the set of resources. That is, $X_i = M$ for every player i. The sets S_i will have no use, and therefore we ignore them. The set of types of i is $T_i = \{\emptyset, M\}$. The signaling functions are defined as follows: $\tilde{t}_i(J) = M$ if $i \in K$ and $\tilde{t}_i(J) = \emptyset$ if $i \notin J$. Recall that $Z_i = M \cup \{e_i\}$, where e_i denotes a dummy action. The cost function of i is $c_i : \Omega \times Z \to \Re$, where for $i \in J \subseteq N$, $c_i(J, z) = w_{z_i}(\sigma_{z_i}(z))$, where $\sigma_{z_i}(z)$ is the number of all players $l \in N$ for which $z_l = z_i$. The values of $c_i(J, z)$ for subsets of N that do not contain i are not relevant and should not be specified. The above pre-Bayesian game is finite. We denote its mixed extension by $G_\Gamma(n)$, and we referred to $G_\Gamma(n)$ as a *resource selection game with unknown number of players*. A strategy of player i in $G_\Gamma(n)$ can be described by a mixed action $q[i] \in \Delta(M)$. That is, when receiving the signal M, i uses $q[i]$ and when receiving the signal $\emptyset$, i uses her dummy action, e_i. Since we deal with costs and not with payoffs we use minimax rather than maximin in the definition of safety-level equilibrium. let $\mu = q[1], \cdots, q[n])$ be a strategy profile in $G_\Gamma(n)$, and let i be a player. Let $\mu[-i] = (q[l])_{l \in N \setminus \{i\}}$ be the profile of strategies of the other players. If i is active, that is she received the signal M, then the set of states that are compatible with i's signal, $\Omega_i(M)$ is the set of all nonempty subsets of N that contain i. If all cost functions are non-decreasing, and if i believes that all other players are using the profile $q[-i]$, then it is obvious that the worst case scenario for i is obtained in the state N. Thus we have:

Lemma 4 *Let Γ be a resource selection system in which the resource cost functions are non-decreasing. Let $\mu \in \Delta(M)^n$. μ is a safety-level equilibrium in $G_\Gamma(n)$ if and only if μ is a mixed-action equilibrium in $\Gamma(n)$.*

Proof: Assume μ is a mixed-action equilibrium in $\Gamma(n)$. Let i be an active player. By the comment we made before the statement of the lemma,

$$\min_{p[i]\in\Delta(M)} \max_{S\in\Omega(M)} c_i(J, p[i], \mu[-i]) =$$

$$\min_{p[i]\in\Delta(M)} c_i(N, p[i], \mu[-i]).$$

Because μ is a mixed-action equilibrium in $\Gamma(n)$, the $\min$ in the right hand-side of the above formula is attained at $q[i]$. Therefore μ is a safety-level equilibrium in $G_\Gamma(n)$. An analogous argument proves the if part of the lemma. ▮

Theorem 5 *Let Γ be a resource selection system in which the resource cost functions are non-decreasing. $G_\Gamma(n)$ has a unique symmetric safety-level equilibrium. In this symmetric safety-level equilibrium every player is using the strategy p^n, where p^n is the unique symmetric-equilibrium action in $\Gamma(n)$.*

Proof: The proof follows directly from Theorem 1 and Lemma 4. ▮

By Theorem 5, each of the players in $G_\Gamma(n)$ is using the strategy p^n, where p^n is the unique symmetric-equilibrium mixed action in $\Gamma(n)$. However, the cost of each active player in $G_\Gamma(n)$ is not $c^n = c^n(p^n)$, it depends on the true state. If the true state is J, that is J is the set of active players, and $|J| = k$, the cost of each active player i is $c^k(p^n)$. It is worthy to compare this cost with the cost $c^k = c^k(p^k)$ that every player in J would have paid had the players in J known the state. We make these comparison in linear models. We say that a resource selection system is *linear* if for every resource j there exists a constant d_j such that $w_j(k) = w_j(1) + (k-1)d_j$ for every $k \geq 1$. For every number of players, n, the associated resource selection game, $\Gamma(n)$, as well as the associated resource selection game with unknown player set, $G_\Gamma(n)$ will be called *linear* too. Note that in a linear system, w_j is increasing if and only if $d_j > 0$, and w_j is non-decreasing if and only if $d_j \geq 0$. The proof of the following theorem, Theorem 6 is given in the full version:

Theorem 6 *Let Γ be a linear resource selection system with increasing resource cost functions. For every $k \geq 2$ let p^k be the unique symmetric equilibrium in $\Gamma(k)$. There exist an integer $K = K(\Gamma)$, $K \geq 2$ such that for all $n > k \geq K$:*

1. *$c^k(p^k) \geq c^k(p^n)$.*
2. *All inequalities above are strict if and only if there exists $j_1, j_2 \in M$ such that $w_{j_2}(1) \neq w_{j_1}(1)$*

Theorem 6 is applicable to a mechanism design setup in which the organizer knows the number of active players, and the players do not know this number. If the goal of the organizer is to maximize revenue then he is better off revealing his private information. If his goal is to maximize social surplus, then he should not reveal that information. In order to estimate the gain of the players resulting from their ignorance we analyze the function $c^k(p^n)$. The proof of the following theorem is given in the full version:

Theorem 7 *Let Γ be a linear resource selection system with increasing resource cost functions.There exist K such that for every $n \geq K$ the following assertions hold:*

1. *$p_j^n = \frac{n-1+B-w_j(1)A}{Ad_j(n-1)}$, where $A = \sum_{j=1}^m \frac{1}{d_j}$, and $B = \sum_{j=1}^m \frac{w_j(1)}{d_j}$.*
2. *The minimal social cost in $\Gamma(k)$ attained with symmetric mixed-action profiles is attained at p^{2k-1}. Consequently, $c^k(p^n)$ is minimized at $n = 2k-1$.*

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