Path Auction Games When an Agent Can Own Multiple Edges *

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Abstract

We study path auction games in which multiple edges may be owned by the same agent in this paper. The edge costs and the set of edges owned by the same agent are privately known to the owner of the edge. We show that in this setting, given the assumption the losing agent always has 0 payoff, there is no *individual rational* strategyproof mechanism in which only edge costs are reported. If the agents are asked to report costs as well as ownership, we show that there is no efficient mechanism that is *false-name proof*. We then study a first-price path auction in this model. We show that, in the special case of parallel-path graphs, there is always a pure-strategy ϵ-Nash equilibrium in bids. We show that this result does not extend to general graphs: we construct a graph in which there is no such ϵ-Nash equilibrium.

1 Introduction and Motivation

In the *path auction game*, there is a network $G = (V, E)$, in which each edge $e \in E$ is owned by an agent. The true cost of e is private information and known only to the owner. Given two vertices, source s and destination t, the customer's task is to buy a path from s to t. This path auction can be used to model problems in supply chain management, transportation management, QoS routing and other domains. Recently, path auctions have been extensively studied [11, 9, 2, 8, 4]; much of this literature has focused on the Vickrey-Clarke-Groves (VCG) mechanism [12, 3, 6]. In the VCG mechanism, the customer pays each agent on the winning path an amount equal to the highest bid with which the agent would still be on the winning path.This mechanism is attractive because it is efficient and*strategyproof,i.e.,* the dominant strategy for each agent is to report its true cost.

In the traditional path auction model, each agent only owns one edge in the graph, and there is no cooperation between agents. Here, we study a variant of the path auction game in which *each agent may own multiple edges*. In this extended model, if the ownership information is publicly available (i.e. the customer knows which agent owns which edge), the VCG mechanism design approach yields a strategyproof mechanism.

In practice, however, the ownership information is more likely to be private – it could be costly for the customer to find out the true ownership information, or the agent may have an incentive to hide its true ownership information in order to get better payoff. For example, in Figure 1, there are two agents: a and b. Agent a owns edges (s, i) and (i, t) with true cost 1 each; agent b owns edges (s, j) and (j, t) with true cost 2 each. If agents a and b reveal the true ownership information to the customer, the most natural VCG mechanism will choose path (s, i), (i, t) as the winning path and pay agent a an amount equal to 2. However, if agent a hides its ownership information, the mechanism will treat edges (s, i) and (i, t) as owned by different agents. When the agents bid their true costs, the winning path stays the same, but the payment to agent a would be $2 \times 3 = 6$. Moreover, when the ownership information is not available to the customer, agent

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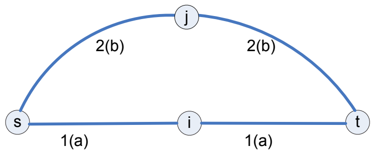

Figure 1: VCG mechanism is not strategyproof for this game

a can increase its payoff by bidding lower than its true cost. For example, it can bid 0.5 for both edges (s, i) and (i, t). This does not change the winning path, but the payoff to agent a would increase to $2 \times 3.5 = 7$. Hence, the straightforward VCG mechanism, which assumes that each edge is owned by an individual agent, is not strategyproof. In this paper, we model situations in which each agent can own multiple edges at the same time, but the ownership information is private. Thus the traditional path auction model is a special case of our extended model. One real-life example of our model is an online auction system in which each seller/buyer can have multiple accounts in the system. Now, if a buyer wants some combination of goods that can be expressed in path auction form, it is hard for her to find the true identity of each seller account, and so she is faced with the unknown-ownership scenario.

In this paper, we analyze path auctions under two solution concepts: dominant strategies and Nash equilibrium in bids. We begin by studying *truthful* dominant strategy mechanisms, *i.e.* strategyproof mechanisms. We show that if the agents only submit bid prices in the auction for each edge, there is no strategyproof mechanism that satisfies individual rationality under assumption that the losing agent always has 0 payoff. The natural extension is to consider mechanisms in which agents are invited to reveal their entire private information, the ownership of edges as well as the costs. An important strategic property in this setting is that the mechanism is *false-name proof* [13], *i.e.*, an agent cannot gain by dividing her owned edges among two or more pseodonyms. We show that earlier results on false-name proof mechanisms [13] imply that there is no Pareto-efficient false-name proof mechanism in the extended auction format.

We next turn to a first-price auction bidding game, and study ϵ-Nash equilibria of this game. For the class of parallel-path graphs, we constructively prove that at least one ϵ-Nash equilibrium exists, and we prove a lower bound on the total payments in any such equilibrium. However, we find a non-parallel-path graph which can be proven not to have a pure strategy ϵ-Nash equilibrium.

Please note that all proofs have been deferred to the appendix.

1.1 Related work

Path auction games have been extensively studied in recent years. Nisan and Ronen introduced the shortest-path game in their paper on algorithmic mechanism design [11], and showed that the VCG mechanism for this problem is computationally tractable. Hershberger and Suri [7] described an improved algorithm for this problem. However, several authors have noted that the VCG mechanism may pay much higher than the true cost of the winning path. This has led to the study of the frugality [2] of VCG mechanism. Archer and Tardos [2], and Elkind *et. al* [4] studied the frugal path auction mechanism, and showed that the payments can be arbitrarily high. Karlin and Kempe [9] extended the path model to a more general set system model and introduced a new frugality ratio definition; they designed a mechanism that performs better than VCG in path auction. The problem of agents owning multiple edges was mentioned as future work in [9]. Immorlica *et. al.* [8] studied first-price path auctions in the traditional single-ownership setting. They showed the existence of a strong ϵ-Nash equilibrium in bids, and bounded the payments in equilibrium. Yokoo *et. al.* [13] introduced the concept of false-name proof mechanisms, and showed that in combinatorial auctions, there is no false-name proof mechanism that satisfies Pareto efficiency.

2 Definitions and Problem Statement

First, we introduce the formal definition of path auction based on the definition of set system in [9].

The simple model of path auction: Given a graph $G = (V, E)$, each edge $e \in E$ is owned by an agent and has a cost c_e, the true cost it incurs if it is selected. This value is private, i.e. known only to the agent which owns e. We define the feasible set $F = \bigcup_i P^i(s, t)$, where $P^i(s, t)$ is the ith path from s to t. Given two specific vertices s(the source) and t(the destination), the task of the customer is to buy a path from s to t by auction. It consists of the following two steps:

1. Each agent submits a sealed bid b_e to the customer. The bidding vector b is $(b_{e_1}, b_{e_2}, ..., b_{e_m})$, where b_{e_i} is the bidding price for edge $e_i \in E$. Moreover, let B denote the bidding space that is the set of all possible bidding vectors.

2. Given the bidding vector b, the customer selects a path P^i from the feasible set F as the winning path, and computes a payment $p_e \geq b_e$ for each edge $e \in P^i$. We say that if an agent owns an edge e on the winning path P^i, it *wins*, and all other agents *lose*.

In order to implement the auction, we need to design a mechanism $(f, p_1, p_2, ..., p_m)$, where $f : B \rightarrow F$ selects one element in the feasible set as the winning path and $p_i : B \rightarrow R$ computes the payment to agent i. Moreover, we assume that:

- (G, F) is common knowledge to the customer and all agents.
- the game is *monopoly free*, which means no edge is in all feasible sets, i.e. $\bigcap_{P^i(s,t) \in F} P^i(s,t) = \varnothing$.
- the agent is rational and has quasilinear utility, i.e., the agent want to maximize its utility: $u_e = p_e(b) - c_e$ if e is on the winning path; or else $u_e = 0$.

Definition 1. A mechanism is *strategyproof* if, for any agent i that owns edge e, any $b_{-i} \in B_{-i}$ and $b_i^{'}$, $p_i(c_e, b_{-i}) - c_e \geq p_i(b_i^{'}, b_{-i}) - c_e$, where b_{-i} is the bidding vector of all agents except i.

The VCG mechanism is strategyproof in the simple path auction game, *i.e.* the dominant strategy of each agent is to bid its true cost in VCG mechanism.

In the simple path auction model, each agent only owns one edge in the graph. We extend the model in the following way:

The extended model of path auction: Now assume that each agent can own multiple edges. We can partition the edge set E as: $E = \bigcup_i E_i$, where E_i is the set of edges owned by agent i. We also assume that if agent i owns k edges, *i.e.* $|E_i| = k$, it has k identities $ID_i = \{ID_{i1}, ID_{i2} ..., ID_{ik}\}$, one for each edge to use in the auction. In the extended model, a game is monopoly free if for any agent i, there is at least one path between s and t in graph $(V, E \setminus E_i)$. A mechanism is strategyproof if for any agent, the dominant strategy is to bid the true cost for each edge it owns. Moreover let p_i denote the payment to agent i. Then p_i is equal to $\sum_{e_j \in E_i} p_{e_j}$. According to the type of bidding space, we can define two types of auctions:

Path Auction of Type I: In this type of auction, the agent is only asked to submit the bidding price for each edge it owns. The mechanism will select the winning path and compute the payment to each edge.

Path Auction of Type II: In this type of auction, the agent is asked to submit the ownership information about which set of edges it owns and the bidding price for each edge it claims to own. Let $o = (o_{e_1}, o_{e_2}, ..., o_{e_m})$ be the claimed ownership information vector, where if edge e_j is owned by agent i (*i.e.* $e_j \in E_i$), $o_{e_j} \in ID_i$. We assume that no more than one agent claims to own the same edge and each edge is claimed to be owned by some agent. Since the agent has one identity for each edge it owns, it can choose arbitrary strategy to report the ownership information for edges owned by itself.

We will not only study the strategyproof mechanism in the above two types of auctions, but also a weaker solution concept: ϵ-Nash equilibrium.

Definition 2. An ϵ-Nash equilibrium for a game is a set of strategies, one for each player, such that no player can unilaterally deviate in a way that improves its payoff by at least ϵ.

3 The Nonexistence of strategyproof Mechanism

In the extended model of path auction, the question to answer is: Is it possible to design a mechanism such that it is in every agent's best interest to bid her true cost? We focus on the auction of type I in subsection 3.1 and the auction of type II in subsection 3.2.

3.1 No individual rational strategyproof mechanism in auction of type I

In auction of type I, we can construct a trivial strategyproof mechanism, which always selects a fixed path as the winning path and pays a fixed amount of money to the edges on the path. We call such a mechanism the *dictator mechanism*. It is not hard to verify that the dictator mechanism is strategyproof, but it might not be *individual rational*. The definition of individual rational is:

Definition 3. A mechanism is *individual rational* if, for any agent i, the payment to itself is at least the true incurred cost when it is selected by the mechanism, *i.e.* $p_i \geq c_i$.

Based on the definition of individual rationality, we have the following theorem:

Theorem 1. *Given the assumption that the losing agent always has 0 payoff, there is no strategyproof mechanism for auction of type I that satisfies individual rationality.*

We believe that if we remove the assumption that the losing agent always has 0 payoff, the theorem still holds. It would be interesting to find a simple proof for such extension of theorem 1.

3.2 No false-name proof mechanism in auction of type II that satisfies Pareto efficiency

As shown in previous subsection, if the agent only submits the bidding price information, it is almost impossible to enforce the agent to bid its true cost. In order to make the agent bid truthfully, the customer may ask the agents to reveal more information, such as the ownership information, besides the bidding price information. Therefore we consider auction of type II. First we give the definition of false-name proof mechanism [13] in the context of path auction game.

Definition 4. A mechanism is false-name proof if for any fixed bidding vector b_{-i} and the claimed ownership vector o_{-i} by all agents other than i, it is agent i's best interest to bid the true cost of each edge it owns, *i.e.* $b_i = (c_{e_{i1}}, c_{e_{i2}}, ..., c_{e_{ik}})$ where $E_i = \{e_{i1}, e_{i2}, ..., e_{ik}\}$, and to claim the real ownership information $o_i = \underbrace{(ID_{ij}, ID_{ij}, ..., ID_{ij})}_{k}$ where $1 \leq j \leq k$.

For situations in which the true ownership cannot be determined, a false-name false-name proof mechanism [13] is desirable. The next natural question is: Is it possible to design a false-name proof mechanism in the extended model of path auction game? Yokoo *et. al.* [13] showed the following impossibility result for combinatorial auctions:

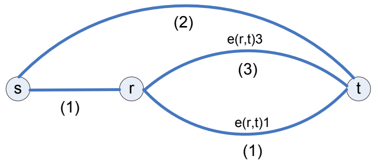

Figure 2: There is no false-name proof mechanism which satisfies Pareto efficiency in this game.

Proposition 1. [13] In combinatorial auctions, there is no false-name proof auction protocol that satisfies Pareto efficiency. □

The definition of Pareto efficiency is:

Definition 5. A winning path selection mechanism is *Pareto efficient* if given the winning path $P^i(s,t)$, $\forall k$, $\sum_{e \in P^i(s,t)} c_e \leq \sum_{e \in P^k(s,t)} c_e$, which means that the mechanism always selects the path from s to t with minimum cost.

The above proposition is proved by constructing a generic counter example. Since path auction is only an instance of more general class of combinatorial auctions, it might be possible to design a false-name proof mechanism for path auction even the impossibility result holds for combinatorial auctions. However, the generic counter example constructed in [13] can be easily transformed to a path auction game and show the impossibility result in auction of type II.

Proposition 2. In the extended model of path auction game, there is no false-name proof mechanism for auction of type II that satisfies Pareto efficiency.

4 Existence of ϵ-Nash Equilibrium

Since strategyproof mechanism is not widely achievable in the extended model of path auction game, we need to extend the solution concept from dominant strategies to non dominant strategies. The concept of ϵ-Nash equilibrium is an important candidate. In this section, we study the existence of ϵ-Nash equilibrium under the VCG mechanism and the *first-price auction* mechanism [8], which elicits the bids from the agents, chooses the cheapest path respect to the bidding vector as the winning path and pays each winning agent exactly the bidding price.

Since VCG is not strategyproof in the extended model, a natural question to ask is: If we apply VCG mechanism, is there an equilibrium in the resulting game? For the game in Figure 1, suppose b is the bidding vector that reaches an ϵ-Nash equilibrium. As the straightforward VCG mechanism assumes each edge is owned by an individual agent, whatever the winning path is in Figure 1, the winning agent can increase its payoff by decreasing its bidding vector until its bidding prices reach 0. This implies that the winning agents have the incentive to bid as low as they can if all other agents bid truthfully. We will exclude such equilibrium from discussion.

Now, we would like to study first price auction mechanism in the extended model. In practice, a rational agent is not willing to bid below the true cost for each edge in first price auction because such strategy may incur negative payoff to the agent. Therefore, we assume that the bidding price of each edge is at least its true cost, *i.e.* $\forall e, b_e \geq c_e$, when we discuss ϵ-Nash equilibrium in the following. In the next, we would like to show the existence of ϵ-Nash equilibrium in the *parallel-path graph* [5], which can be defined inductively as:

Definition 6. A parallel-path graph(PPG) is a network (V, E, s, t), such that one of the following conditions is satisfied:

Base Case: A path from s to t is a PPG;

Parallel: Suppose $G_1 = (V_1, E_1, s, t)$ and $G_2 = (V_2, E_2, s, t)$ are PPG such that $V_1 \bigcap V_2 = \varnothing$ and $E_1 \bigcap E_2 = \varnothing$. Set $V = V_1 \bigcup V_2$ and $E = E_1 \bigcup E_2$, then (V, E, s, t) is a PPG.

Given the definition of parallel-path graph, we can prove the following theorem:

Theorem 2. *If the underlying network is a parallel-path graph, the first-price path auction has an ϵ-Nash equilibrium.*

Since the underlying network (V, E, s, t) is a parallel-path graph, we can represent it as $\bigcup_k P^k(s,t)$, where $P^k(s,t)$ is the kth path from s to t and $\forall i \neq j$, $P^j(s,t) \bigcap P^j(s,t) = \varnothing$. Moreover, let $C(P^k(s,t)) = \sum_{e \in P^k(s,t)} c_e$ denote the cost of path $P^k(s,t)$ with respect to true cost vector c. We sort the paths from low to high according to their costs, i.e. the path with lower cost has smaller index. If agent A_i owns at least one edge on the cheapest path $P^1(s,t)$, let $LPI(A_i)$ be the smallest path index such that path $P^{LPI(A_i)}(s,t)$ does not have an edge owned by agent A_i but for any path that has smaller path index than $LPI(A_i)$, it must have at least one edge owned by agent A_i. We compute $LPI(A_i)$ for each agent A_i that owns at least one edge on $P^1(s,t)$. Suppose agent A_k has the highest value: $LPI(A_k)$ (break ties arbitrarily), we can bound the payment of any ϵ-Nash equilibrium in the following corollary, which is derived directly from the proof of theorem 2.

Corollary 1. The total payment in any ϵ-Nash equilibrium is at least: $C(P^{LPI(A_k)}(s,t))$.

The lower bound given in corollary 1 is tight. In order to study the frugality of first price auction mechanism in our model, an interesting question is to find out the upper bound of the payment in any ϵ-Nash equilibrium for parallel-path graph. A small value upper bound will imply that first-price auction mechanism is frugal in our model.

Although, there exists an ϵ-Nash equilibrium for parallel-path graph, we can find a non-parallel-path graph that does not have a pure-strategy ϵ-Nash equilibrium. We show this counter example in Figure 3 and the following proposition proves this result. Please note that in Figure 3, the integer number in the bracket denotes the identity of the agent who owns that edge.

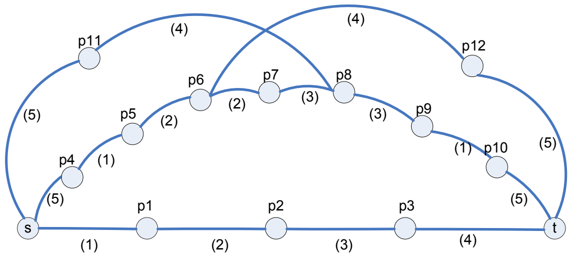

Figure 3: There is no pure-strategy ϵ-Nash equilibrium in this first-price path auction game.

Proposition 3. Given the assumption that each edge's bidding price is at least its true cost, i.e. $\forall e \in E, b_e \geq c_e$, the graph showed in Figure 3 can not have a pure-strategy ϵ-Nash equilibrium in first-price path auction.

5 Conclusion And Future Work

In this paper, we studied the path auction games in which an agent can own multiple edges. Our model is more general than the simple path auction model.

However, our results show that strategyproofness is not widely achievable in the extended model; moreover, general graphs may not have a pure-strategy ϵ-Nash equilibrium in first-price path auction mechanism. Therefore, our model leaves a few challenges.

In this paper, although we have found an ϵ-Nash equilibrium for parallel-path graph, we do not have a mechanism such that when the agents play the game under the mechanism, they can reach the ϵ-Nash equilibrium. So a natural open problem is to design such a mechanism. Moreover, we believe that there exists an ϵ-Nash equilibrium for series parallel-graph [5]. It would be interesting to extend the result of theorem 2 to more general class of graphs.

For the non-parallel-path graphs, we found a counter example which does not have a pure-strategy ϵ-Nash equilibrium in first price path auction mechanism. An interesting question is to analyze the mixed strategy Nash equilibrium or Bayes-Nash equilibrium given some distribution on the edge costs.

Acknowledgements

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APPENDIX

Theorem 3. *[9, 1, 10] A strategyproof mechanism has the following two properties.*

1. *A mechanism is strategyproof only if the selection rule is monotone: No losing agent can become a winner by rasing his bid, given fixed bids by all other agents.*

2. *Given a monotone selection rule, there is a unique strategyproof mechanism with this selection rule. This mechanism pays each agent his threshold bid, i.e. the highest value he could have bid and won.*

Lemma 1. *In the extended path auction model, given the assumption that the losing agent always has 0 payoff, for any individual rational strategyproof mechanism* $(f, p_{e_1}, ..., p_{e_m})$ *and a given* *strictly positive bidding vector* $b^1 = (b^1_{e_1}, ..., b^1_{e_m})$, *we can construct another bidding vector* $b' = (b'_{e_1}, ..., b'_{e_m})$ *such that when the agents bid according to* b', *all the edges on the winning path have positive payoffs. Moreover,* $\forall j, |b^1_{e_j} - b'_{e_j}| \leq \epsilon$, *where* ϵ *is a small positive number.*

Proof: Suppose initially when the agents bid according to b^1, the winning path is P^1, *i.e.* $f(b^1) = P^1$. Moreover, we assume that all the losing agents have payoff zero.

In the first round, for an edge $e_1 \in P^1$, we decrease the bidding price of it from $b^1_{e_1}$ to $b^1_{e_1} - \epsilon$ and keep the biding prices of all other edges unchanged. Let $\mathcal{TWP} = \{e_1\}$. Thus we get a new bidding vector b^2 and $f(b^2) = P^2$. According to theorem 3, e_1 must be on the new winning path P^2. Moreover, the payment to edge e_1 should not change, *i.e.* $p_{e_1}(b^1) = p_{e_1}(b^2)$. Or else, if $p_{e_1}(b^1) > p_{e_1}(b^2)$, when the true cost of edge e_1 is $b^1_{e_1} - \epsilon$, edge e_1 can improve its payoff by increasing its bidding price to $b^1_{e_1}$. If $p_{e_1}(b^1) < p_{e_1}(b^2)$, when the true cost of edge e_1 is $b^1_{e_1}$, edge e_1 can improve its payoff by decreasing its bidding price to $b^1_{e_1} - \epsilon$. Both cases contradict to the strategyproofness. Since $p_{e_1}(b^1) = p_{e_1}(b^2)$, when edge e_1 decreases its bidding price from $b^1_{e_1}$ to $b^1_{e_1} - \epsilon$, its payoff will increase by ϵ.

In the kth round where $k \geq 2$, for an edge $e_k \in P^k$ but $e_k \notin \mathcal{TWP}$, we decrease the bidding price of it from $b^k_{e_k}$ to $b^k_{e_k} - \frac{\epsilon}{2^{k-1}}$ and keep the biding prices of all other edges unchanged. Thus we get a new bidding vector b^{k+1} and $f(b^{k+1}) = P^{k+1}$. Let $\mathcal{TWP} = \mathcal{TWP} \bigcup \{e_k\}$. Similar to the above arguments, edge e_k must be on the new winning path P^{k+1} and its payoff is increased by $\frac{\epsilon}{2^{k-1}}$ because the payment to it does not change. Moreover, any edge $e_j \in \mathcal{TWP}$ is still on the new winning path P^{k+1} and its payoff cannot decrease by more than $\frac{\epsilon}{2^{k-1}}$, *i.e.* $p_{e_j}(b^{k+1}) - p_{e_j}(b^k) \geq -\frac{\epsilon}{2^{k-1}}$. If $p_{e_j}(b^{k+1}) - p_{e_j}(b^k) < -\frac{\epsilon}{2^{k-1}}$, when an agent i owns both edges e_j and e_k, and the true cost of edge e_k is $b^k_{e_k} - \frac{\epsilon}{2^{k-1}}$, agent i can increase its payoff by increasing the bidding price of edge e_k to $b^k_{e_k}$. This contradicts to the strategyproofness. Since the payoff of edge e_j cannot decrease by more than $\frac{\epsilon}{2^{k-1}}$ and for any k and a finite number $N > k$, $\frac{\epsilon}{2^k} > \sum_{i=k+1}^{N} \frac{\epsilon}{2^i}$, it implies that edge e_j is still on the new winning path P^{k+1} and it has positive payoff.

Finally, when the process is terminated, the winning path $P = \mathcal{TWP}$ and the final bidding vector is b'. According to the argument above, all the edges on the winning path must have positive payoffs when the agents bid according to b'. □

Theorem 1. *Given the assumption that the losing agent always has 0 payoff, there is no strategyproof mechanism for auction of type I that satisfies individual rationality.*

Proof: According to lemma 1, for any individual rational strategyproof mechanism $(f, p_{e_1}, ..., p_{e_m})$, we can construct a sequence of bidding vectors $b(r) = (r \times b_{e_1} - \epsilon(e_1, r), ..., r \times b_{e_m} - \epsilon(e_m, r))$ such that the winning agents always have positive payoffs, where $r \in \mathcal{N}$ and $\forall j, \epsilon(e_j, r)$ is a small positive number. Let $\mathcal{PB} = \{b(r) | r \in \mathcal{N}\}$ denote the set of all such bidding vectors. For each $b(r) \in \mathcal{PB}$, $f(b(r))$ will select a winning path. Since there are infinite number of $b(r)$s, but only finite number of possible winning paths, there must be an infinite subsequence $\mathcal{SPB} = \{b(r_1), b(r_2)......\}$ such that $\forall b(r_i) \in \mathcal{SPB}$, $f(b(r_i)) = P$ always selects P as the winning path. According to the assumption of individual rationality, and given that the payment to each edge is finite, we can find two bidding vectors $b(r_p), b(r_q) \in \mathcal{SPB}$ such that for any edge e_j on the winning path P, $b(r_p)_{e_j} \leq p_{e_j}(b(r_p)) < b(r_q)_{e_j} \leq p_{e_j}(b(r_q))$.

Given a bidding vector b such that the losing agent has payoff 0 while the winning agent has positive payoff. If for an edge e_j not on P, increasing b_{e_j} to b'_{e_j} can change the winning path from P to P' *i.e.* there exists an edge $e \in P$ but $e \notin P'$, we can get some contradiction. According to theorem 3, e_j cannot be on P'. Thus its payoff is still 0. Assume an agent i owns both e_j and e. Since e has positive payoff when it is on winning path P, if the true cost of e_j is b'_{e_j}, agent i can increase its payoff by understating e_j's true cost as b_{e_j}. This contradicts to the strategyproofness. W.O.L.G. we assume that the winning path is a simple path, thus $P = P'$. Moreover, increasing b_{e_j} does not change the payment to any edge. For any edge $e \notin P$, the payment to it is always 0. Suppose increasing b_{e_j} to b'_{e_j} can increase the payment to edge $e \in P$ from p_e to p'_e, *i.e.* $p_e < p'_e$. When an agent i owns both e_j and e, and the true cost of edge e_j is b_{e_j}, agent i can increase its payoff by overstating e_j's cost as b'_{e_j}. This contradicts to the strategyproofness. Similarly, we can get the contradiction when increasing b_{e_j} to b'_{e_j} can decrease the payment to edge $e \in P$.

Similarly, we can prove that for an edge e_j on the winning path P, decreasing b_{e_j} cannot change either the winning path or the payment to any edge in the graph.

When we have bidding vectors $b(r_p)$ and $b(r_q)$, we can construct a new bidding vector b^* such that $b^*_{e_j} = \min\{b(r_p)_{e_j}, b(r_q)_{e_j}\}$ if e_j is on the wining path P while $b^*_{e_j} = \max\{b(r_p)_{e_j}, b(r_q)_{e_j}\}$ if e_j is not on the wining path. According to the construction and the above arguments, we can get $\forall e_j$, $p_{e_j}(b(r_p)) = p_{e_j}(b^*) = p_{e_j}(b(r_q))$. This contradicts to the fact that any edge e_j on the winning path P, $p_{e_j}(b(r_p)) < b(r_q)_{e_j} \le p_{e_j}(b(r_q))$.

Therefore, given the assumption that the losing agent always has 0 payoff, there is no strategyproof mechanism for auction of type I that satisfies individual rationality. □

Proposition 2. *In the extended model of path auction game, there is no false-name proof mechanism for auction of type II that satisfies Pareto efficiency.*

Proof Sketch: We are going to prove this proposition by presenting a generic counter example as [13] assuming there is an efficient false-name-proof mechanism. The generic counter example is given in Figure 2. In this example, edges $e(s,r), e(r,t)_1$ are owned by agent 1, edge $e(s,t)$ is owned by agent 2 while edge $e(r,t)_3$ is owned by agent 3. Since agent 1 owns two edges, when bidding in auction of type II, the ownership of edge $e(r,t)_1$ can be claimed as agent 1 or some artificial non existent agent 4. Given this example, the proof of this proposition is almost the same as the proof of Proposition 1 in [13]. □

Theorem 2. *If the underlying network is a parallel-path graph, the first-price path auction has an ϵ-Nash equilibrium.*

Proof Sketch: The ϵ-Nash equilibrium bidding vector is constructed as follows. Suppose the parallel-path graph is $(V, E, s, t) = \bigcup_k P^k(s,t)$, where $P^k(s,t)$ is the kth path from s to t and $\forall i \neq j$, $P^j(s,t) \bigcap P^j(s,t) = \varnothing$. Initially, each agent bids its true cost *i.e.* $b = c$. Let $W_b(P^k(s,t)) = \sum_{e \in P^k(s,t)} b_e$ denote the cost of path $P^k(s,t)$ with respect to the bidding vector b. Moreover, we assume that agent A_k has the highest value of $LPI(A_k)$ which is defined before. In order to find the ϵ-Nash equilibrium bidding vector, first we would pick one edge in $E_{A_k} \bigcap P^1(s,t)$ and increase its bidding price until $W_{b'}(P^1(s,t)) = W_b(P^{LPI(A_k)}(s,t)) - \epsilon$, where b' is the new bidding vector. For any path $j \in [2...LPI(A_k)-1]$, we pick one edge in $E_{A_k} \bigcap P^j(s,t)$ and increase its bidding price until $W_{b'}(P^j(s,t)) = W_b(P^{LPI(A_k)}(s,t))$. We call the final bidding vector b^f.

It is not hard to verify that b^f is an ϵ-Nash equilibrium bidding vector. □

Proposition 3. *Given the assumption that each edge's bidding price is at least its true cost, i.e. $\forall e \in E, b_e \ge c_e$, the graph showed in Figure 3 can not have a pure-strategy ϵ-Nash equilibrium in first-price auction.*

Proof: In Figure 3, the numbers in the brackets represent the identities of agents that owns the edges. There are 5 agents in this game and 5 paths from s to t:

Path 1: (s, p_1, p_2, p_3, t)

Path 2: $(s, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, t)$

Path 3: $(s, p_4, p_5, p_6, p_{12}, t)$

Path 4: $(s, p_{11}, p_8, p_9, p_{10}, t)$

Path 5: $(s, p_{11}, p_8, p_6, p_7, p_6, p_{12}, t)$

Let b be the ϵ-Nash equilibrium bidding vector. We claim that the cost of each path respect to b can differ at most by ϵ. We prove this by contradiction. Suppose P^k is the winning path and $\exists j, W_b(P^j) > W_b(P^k) + \epsilon$. For any agent $i \in [1...5]$ in Figure 3, there is only one path that does not have edges owned by i. We can assume that path P^j does not have edges owned by the agent i, but for all other 4 paths, agent i owns edges on all of them. Thus agent i can increase the bidding prices of its edges(but still keep P^k as the winning path) such that its payoff can increase by at least ϵ. Then contradiction occurs. Therefore, if b is an ϵ-Nash equilibrium bidding vector, the cost of each path respect to b can differ at most by ϵ. Based on this fact, we can get the following two equations:

$|(b_{p_8,p_7} + b_{p_7,p_6} + b_{p_6,p_{12}} + b_{p_{12},t}) - (b_{p_8,p_9} + b_{p_9,p_{10}} + b_{p_{10},t})| \le \epsilon ...(1)$

$|(b_{p_6,p_7} + b_{p_7,p_8} + b_{p_8,p_9} + b_{p_9,p_{10}} + b_{p_{10},t}) - (b_{p_6,p_{12}} + b_{p_{12},t})| \le \epsilon ...(2)$

According to equations (1) and (2), we can get:

$$(b_{p_7,p_8} + b_{p_6,p_7}) + b_{p_8,p_9} + b_{p_9,p_{10}} + b_{p_{10},t} - \epsilon$$

$$\le -(b_{p_7,p_6} + b_{p_8,p_7}) + b_{p_8,p_9} + b_{p_9,p_{10}} + b_{p_{10},t} + \epsilon$$

Therefore, the following inequality holds:

$$b_{p_7,p_6} + b_{p_8,p_7} + b_{p_6,p_7} + b_{p_7,p_8} \le 2\epsilon$$

Moreover, according to our assumption $\forall e, c_e \le b_e$, the following inequality holds

$$c_{p_7,p_6} + c_{p_8,p_7} + c_{p_6,p_7} + c_{p_7,p_8} \le 2\epsilon$$

When ϵ is small enough, but the true cost of each edge is large enough, contradiction occurs. Therefore, there is no ϵ-Nash equilibrium of first-price auction in Figure 3. □