Social Influence and Evolution of Market Share

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ABSTRACT

We propose a model for the evolution of market share in the presence of social influence. We study a simple market in which the individuals arrive sequentially and choose one of the products. Their decision of which product to choose is a stochastic function of the inherent quality of the product and its market share.

Using techniques from stochastic approximation theory, we show that market shares converge to an equilibrium. We also derive the market shares at equilibrium in terms of the level of social influence and the inherent fitness of the products.

In a special case, when the choice model is a multinomial logit model, we show that inequality in the market increases with social influence and with strong enough social influence, monopoly occurs. These results support the observations made by Salganik et. al. [27] in their experimental study of cultural markets.

Categories and Subject Descriptors

J.4 [**Social and Behavioral Sciences**]: Economics; C.4 [**Performance of Systems**]: Modelling techniques

General Terms

Theory, Economics

1. INTRODUCTION

The objective of this paper is to study markets in which the decision of individuals in choosing a product does not depend just on the quality of the product but also its market share. This is observed in many online social settings like Youtube or Digg, in which clips or news items that have a high "market share" or high number of views are chosen more frequently [6].

These markets give rise to very interesting questions. Do they converge to an equilibrium? Is the outcome (or the equilibrium) of this market predictable? Can we observe a significant difference in the outcome if we increase the level of social influence? What is the effect of the inherent appeal, or quality, of the products on their market share?

We consider a market in which individuals sequentially choose one of the available products based on its inherent quality as well as its market share. The market share of a product is defined as the fraction of people who have chosen that product before. Also, the decision of every individual is a stochastic function of these two parameters.

We show that market shares converge to an equilibrium if there is a sufficiently large number of participants. Furthermore, it is possible to derive the equilibrium as a solution of an Ordinary Differential Equation (ODE). The proof of this theorem uses techniques from stochastic approximation theory, which relate the limit behavior of a stochastic process to the limit behavior of a differential equation.

We also study the rate of convergence to the equilibrium. We prove that the difference of the stochastic process and its corresponding ODE, converges weakly to a Gaussian diffusion. In other words, for a sufficiently large number of people in the market, with high probability, the difference between the market share and the equilibrium point is less than $\frac{c}{\sqrt{n}}$, where c depends on the social influence.

The above results are for a very general class of functions and imply some of the existing results on balls and bins processes in [21, 9]. In these models, balls are sequentially thrown into bins so that the probability that a bin with n balls receives the next ball is proportional to $f(n)$, for some feedback function f. The feedback function is of type $f(n) = n^p$ for $p > 0$. It has been shown that when $p > 1$, almost surely there is some bin that gets all but finitely many balls. This is what we refer to as the *monopoly* case. When p is less than 1, the asymptotic fraction of balls in each bin are the same, whereas $p = 1$ becomes the classic Polya Urn model. Our analysis imply these results for $p < 1$ and $p > 1$.

Our final result deals with the case when the choice probability function is logit. As a special case, we focus on the multinomial logit model due to its wide use for modeling social influence in economics literature. There are also many empirical studies that use the logit model to analyze social influence [29, 14, 1, 11, 24].

By focusing on this special case, we can measure the effect of social influence on the market share more directly. For example, we can show that the number of equilibria increases with the social influence. This can be interpreted as the increase of the unpredictability of the market.

We also study the effect of social influence on inequality.

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Our measure of inequality is Gini coefficient as defined in [27]:

$$G = \frac{1}{2M}\sum_{\alpha,\beta} |\phi_\alpha - \phi_\beta|$$

M is the number of products and ϕ_α is the market share of product α. We observe that Gini coefficient is monotone with the social influence. Furthermore, with large enough social influence, monopoly occurs, that is, one of the products gets the largest market share while for the rest, the market shares become almost zero.

Both of these results are consistent with the observations made in Salganik et al. [27].

1.1 Related Literature

The effect of social influence on collective behavior has been studied by extensively. Social learning models and *herding* models, have been incorporated into standard models of economic decision-making by a growing number of theoretical studies.

Initially, the problem was pointed out by [28, 12, 15], where authors started studying the effects of interacting agents. Later it was studied in well-known results of [2, 4, 13]. Quantitative models of herding and interaction effects have also been observed by [22, 10, 7, 20, 23, 26].

One branch of research on herding uses rational learning explanations, grounded in theories of Bayesian learning. In this setting, herding is described as a Bayes rational response to imperfect information. Here, social information that comes from the action of others is used during the Bayesian updating of *a priori* probabilities. These results also establish the presence of a stable solution [8, 4].

The underlying hypothesis in these models is that individuals are rational and different people will, on average, behave in the same way. Our model relaxes this assumption by the affect of the noise term. At the same time, the optimization problem solved by each consumer in our model is not as complicated as the Bayesian model.

Despite many theoretical studies, social effects have been hard to quantify empirically. This is mostly due to the difficulty of drawing inferences from the data as described in detail by [19]. One can still find empirical studies on herding in many different contexts such as crime [14], labor supply [30], stock market participation [17], and choice of health plans [3]. One of the main inspirations for this paper has been the result of Salganik et. al. [27]. In this paper, the authors study a web-based music market with over 14,000 consumers to understand the effect of social influence in cultural markets. They conclude that the inequality and unpredictability in the market increases with social influence.

2. OUR MODEL

We consider a market with m products. At each time step n, a new participant enters the market. After observing the market share of each product, he chooses a product $i \in \{1, ..., m\}$ that satisfies

$$i \in \operatorname*{argmax}_j (Jh(\phi_j(n)) + f_j + \epsilon_j)$$

J is a parameter that measures the social influence, $\phi_j(n)$ is the market share of the product j at time n, h is a monotone smooth function, f_j is the inherent quality of the product j, and ϵ_j is the noise. We call the sequence of market shares, $\phi(n)$, generated according to this model as the *market share sequence*. According to our model, each customer's decision is influenced by what others have chosen before him (the market share), the social influence (for $J > 0$), and the quality of the products. Once the stochastic terms are realized, each customer chooses an alternative that maximizes the above equation. From now on we will use vectors in our notation; $\phi(n) = (\phi_1(n), ...\phi_m(n))$ is the market share vector at time n and $f = (f_1, ...f_m)$ is the quality vector.

The above model can be considered as an instance of the *additive random utility model* in discrete choice theory, which is frequently used by economists [25]. In this model, an agent must choose from a set of alternatives $A = \{1, ..., m\}$ offering some base utility, in our case

$$x_j = Jh(\phi_j(n)) + f_j$$

and some stochastic utility ϵ_j. Each agent chooses the alternative whose total utility is the highest.

We define the *choice probability function* as

$$L : R^m \to P^m$$

where

$$L_i(X) = P(\operatorname*{argmax}_j x_j + \epsilon_j = i)$$

The most common choice probability function used to model herding is the *logit choice function* [29, 14]. If ϵ_j are independent identically distributed random variables with the extreme value distribution, such that

$$F(\epsilon) = exp(-exp(-\eta^{-1}\epsilon - \gamma))$$

where γ is the Euler constant, then the choice probability function is the logit choice function

$$L_i(X) = \frac{e^{\eta^{-1}x_i}}{\sum_j e^{\eta^{-1}x_j}}$$

In our model the n^{th} customer will choose product i with probability

$$C_i^J(\phi(n)) = L_i(Jh(\phi(n)) + f)$$

As a side note, let us give two examples of this function:

Example 1: Let h be the identity function, and $\eta = J^{-1}$. Then,

$$C_i^J(\phi(n)) = \frac{e^{J\phi_i(n)+f_i}}{\sum_j e^{J\phi_j(n)+f_j}}$$

Example 2: Let $h = \ln(x)$ and $\alpha_i = exp(f_i)$. Then our model becomes a generalization of the preferential attachment:

$$C_i^J(\phi(n)) = \frac{\alpha_i(\phi_i(n))^J}{\sum_j \alpha_j(\phi_j(n))^J}$$

For $J = 1$, the above model is used in [5] as preferential attachment with fitness. Also, if fitness of each product is the same, our model becomes the balls in bins process with feedback, where the feedback function is $f(x) = x^J$.

2.1 Main Theorems

Using the above model, we study the behavior of the market shares when a sufficiently large number of customers have visited the market. In the first theorem, we will show that in the general case, the stochastic process of the market share asymptotically follows a deterministic path, which is the solution of an ODE. Next, we will show that the market share converges to an equilibrium at infinity in the special case where h is linear. In the third theorem, we study the rate of convergence of the equilibrium.

THEOREM 1. *Let $\phi(n)$ be the market share at time n. Then, there exists a sequence $(\bar{t}_k)_{k\geq 1}$ such that as $\bar{t}_k \to \infty$*

$$\lim_{k\to\infty} |\phi(k) - \bar{\phi}(\bar{t}_k)| = 0, \tag{1}$$

where $\bar{\phi}$ is the solution of the following ODE:

$$\frac{d}{dt}\bar{\phi}(t) = C(\bar{\phi}(t)) - \bar{\phi}(t). \tag{2}$$

PROOF. For each customer n, define the indicator $I^n \in \mathbb{R}^M$, representing the product selected by customer n. In other words, $I_i^n = 1$ if customer n selects product i, otherwise $I_i^n = 0$.

Suppose product i has been chosen N_i^n times by the first n customers. Let N^n be a vector with m components where component i is N_i^n. Here we assume that each customer buys exactly one product. Therefore,

$$N^{n+1} = N^n + I^{n+1}$$

By definition, market share is $\phi(n) = \frac{1}{n}N^n$ and $C\big(\phi(n)\big) = E[I^n|\phi(n)]$. Hence,

$$\begin{aligned}
\phi(n+1) &= \frac{n}{n+1}\phi(n) + \frac{1}{n+1}I^n \\
E\big[\phi(n+1) - \phi(n)\big|\phi(n)\big] &= -\frac{1}{n+1}\phi(n) + \frac{1}{n+1}E[I^n|\phi(n)] \\
&= -\frac{1}{n+1}\phi(n) + \frac{1}{n+1}C(\phi(n))
\end{aligned}$$

Let $\epsilon_n = \frac{1}{n+1}$, $g(x) = C(x) - x$, and $d_n = I^n - C(\phi(n))$. Therefore,

$$\phi(n+1) = \phi(n) + \frac{1}{n+1}g(\phi(n)) + \epsilon_n d_n \tag{3}$$

So, $\epsilon_n d_n = (\phi(n+1) - \phi(n)) - \epsilon_n g(x)$ is a Martingale difference with respect to the filtration generated by market share. Note that $\sum_{n=1}^{\infty}\epsilon_n = \infty$, and $\sum_{n=1}^{\infty}\epsilon_n^2 < \infty$. Furthermore, C(x) is a probability vector, so $\|g(x)\| \leq 2$ and $\|\phi(n)(\cdot)\|_\infty \leq 1$, and $\|d_n| \leq 2$.

Define $t_0 = 0$ and $t_n = \sum_{i=1}^{n}\frac{1}{i} \approx \log n$ for all n. Consider a continuous time interpolation $\phi^0(\cdot)$ of $\phi(n)$ on $(-\infty, +\infty)$ by:

$$\begin{aligned}
\phi^0(t) &= \phi(0) \qquad \text{for} \qquad \mathrm{t} \leq 0 \\
\phi^0(t) &= \phi(n) \qquad \text{for} \qquad \mathrm{t_n} \leq \mathrm{t} \leq \mathrm{t_{n+1}}
\end{aligned} \tag{4}$$

Also, define the sequence of the shifted process $\phi^n(\cdot)$ by:

$$\phi^n(t) = \phi^0(t_n + t) \qquad \textit{for all} \qquad t \in \big(-\infty, +\infty\big) \tag{5}$$

ODE Method, developed in stochastic approximation [18] gives strong tools to study the behavior of this type of function sequences.

Now we can apply Theorem 2.1 in [18]. This theorem implies that the sequence of functions $(\phi^n(\cdot))$ has a convergent subsequence such as $(\phi^{r_m}(\cdot))$, which converges uniformly to $\bar{\phi}(\cdot)$, the trajectory of the ODE:

$$\frac{d}{dt}\bar{\phi}(t) = C(\bar{\phi}(t)) - \bar{\phi}(t)$$

Consider a sequence δ_m that converges to zero. Then, there exists a subsequence $(\phi^{r_m}(.))$ such that, for all t, we have:

$$\big\|\phi^{r_m}(t) - \bar{\phi}(t)\big\| \leq \delta_m$$

From the definition, we know that $(\phi^{r_m}(t)) = \phi^0(t + t_{r_m}) = \phi(k)$ for some k such that $t_k \leq t_{r_m} + t < t_{k+1}$. So for each m, and large enough k, we have:

$$\big\|\phi(k) - \bar{\phi}(t_k - t_{r_m})\big\| \leq \delta_m$$

Now, we want to show that there exists a sequence $(\bar{t}_k)_{k\geq 1}$ such that as $\bar{t}_k \to \infty$

$$\lim_{k\to\infty} |\phi(k) - \bar{\phi}(\bar{t}_k)| = 0$$

k is bounded by a constant that depends on m, let us show it by $k > u_m$, so now let $f_m = \max(r_m^2 - 1, u_m)$. Then f_m is an increasing sequence going to infinity. For each $f_{m+1} > k \geq f_m$, define $\bar{t}_k = t_k - t_{r_m}$. It is clear that $\bar{t}_k = t_k - t_{r_m} > t_{r_m^2-1} - t_{r_m} > \sum_{i=r_m+1}^{r_m^2}\frac{1}{i} > \ln(r_m)$, so $\bar{t}_k$ goes to infinity as k increases. □

The natural question here is the stability of the above ODE. In the next theorem, we show that if h is a linear function for most of the choice probability functions, the equlibrium is stable. The following theorem also shows that the market share converges almost surely to a finite discrete set of points under some conditions.

THEOREM 2. *Let $C : R^m \to P^m$ be the choice probability function defined in the previous section, where h is the identity function, i.e.*

$$C_i(X) = P(\operatorname*{argmax}_j J\phi_j + f_j + \epsilon_j = i)$$

Also, assume that the random vector ϵ admits a strictly positive density on R^m and is such that C is continuously differentiable. If $\bar{\phi}(t)$ is a trajectory of the ODE

$$\frac{d}{dt}\bar{\phi}(t) = C(\bar{\phi}(t)) - \bar{\phi}(t)$$

then, $\bar{\phi}(t)$ is well-defined over the whole $\mathbb{R}$ and

$$\lim_{t\to\infty}(C(\bar{\phi}(t)) - \bar{\phi}(t)) = 0 \tag{6}$$

PROOF. By *Theorem 2.1* in [16] , there is some convex function $\tilde{W} : R^n \to R$, such that $\nabla\tilde{W} = C$. Now let us define $W : R^n \to R$ as:

$$W(P) = \frac{1}{2}\big(P^{'}P\big) - \tilde{W}(P)$$

It is clear that W for all $P \in \mathbb{R}^m$:

$$\nabla W(P) = \big(P - C(P)\big)^{'}$$

So, the domain of the trajectories of (6) is the whole $(-\infty, +\infty)$. Let $\bar{\phi}(\cdot)$ be the trajectory of the ODE. Then

$$\begin{aligned}\frac{d}{dt}W(\bar{\phi}(t)) &= (\nabla W(\bar{\phi}(t)))(\frac{d\bar{\phi}}{dt}) \\ &= \left(\bar{\phi}(t) - C(\bar{\phi}(t))\right)^{'}\left(C(\bar{\phi}(t)) - \bar{\phi}(t)\right) \\ &= -\left\|\bar{\phi}(t) - C(\bar{\phi}(t))\right\|^2\end{aligned}$$

The above equality holds if $\bar{\phi}(t) \in RP = \{p \in P^m | C(p) = p\}$.

In the next step, we will show that $\|W(X)\|_2$ goes to infinity as X goes to infinity. Let $X = \alpha u$ where $\|u\|_2 = 1$. Clearly, $\|C(X)\|_2 \leq m$. We have:

$$\begin{aligned}\left\|\tilde{W}(X)\right\|_2 &= \left\|\int_0^\alpha d/dt(\tilde{W}(tu))dt\right\|_2 \\ &= \left\|\int_0^\alpha (C(tu)dt).u\right\|_2 \\ &\leq \alpha m \qquad (7)\end{aligned}$$

So $W(X) = 1/2\|X\|_2^2 - \tilde{W}(X)$ goes to $+\infty$ as X goes to infinity.

Since, $W(X)$ is smooth, the minimum of W over $\mathbb{R}^m$ exists and we have $\nabla W(X) = 0$ for all of its minimizers. So, its minimum points belong to RP.

Furthermore, if $C(p) = p$, p is an interior point of the simplex. If $p_i = 0$ for some i, by definition:

$$C_i(p) = P(\operatorname*{argmax}_j(Jp + f_j + \epsilon_j) = i) = 0$$

This contradicts our assumption about the support of ϵ's density. Hence, we can conclude that $W(X)$ is a Lyapunov function, and the trajectory of the ODE converges to a point belonging to the set RP. $\square$

First we showed that the market share converges to an equilibrium in our model. This equilibrium point belongs to the solution of $C(p) = p$, which does not necessarily have a unique solution. When $C(p) = p$ has multiple solutions, the market share can converge to one of these different points. In other words, we have several candidates for the equilibrium point, and the beginning behavior of the market will determine which one of these equilibria will be selected.

So, there is some $\bar{\phi} \in RP = \{p \in P^m | C(p) = p\}$ such that $\phi(n) \to \bar{\phi}$.

Our result also specializes to the *balls in bins process with feedback* studied in [21, 9] as we discussed in example (2). In this model, balls are sequentially thrown into bins so that the probability that a bin with n balls gets the next ball is proportional to $f(n)$, for some feedback function f. The above references focus on the feedback function; $f(n) = n^J$ for J>0 and it has been shown that this family of f has phase transition at $J = 1$: Theorem 2.1 and Theorem 2.2 in [9], states that for each i, $\bar{\phi}_i = \lim_{t\to\infty}\phi_i(t)$ exists. The authors show that when $J > 1$, monopoly occurs, i.e. the fraction of balls in one of the bins goes to one. If $J < 1$, $\bar{\phi}_i = \frac{1}{m}$ for all bins, while for $J = 1$, the limit vector $\bar{\phi}$ is uniformly distributed on the simplex at the same rate. These results are corollaries of our Theorem 1: Let $h(x) = \ln x$ and let the fitness of all products be the same. In this case, the probability choice function is:

$$C_i(\phi) = \frac{\phi_i^J}{\sum_j \phi_j(t)^J},$$

and the ODE in equation (2) has the form:

$$\frac{d}{dt}\phi_i(t) = \frac{\phi_i(t)^J}{\sum_j \phi_j(t)^J} - \phi_i(t).$$

It is easy to see that this dynamic system is stable and the results above follow.

THEOREM 3. *Assume $\phi(n)$ is a* market share sequence*, where $C(\phi)$ is smooth and $\bar{\phi}$ is an isolated stable point of the ODE in equation 2;*

Then, there is some matrix Σ_1 such that

$$U^n = \sqrt{n}(\phi(n) - \bar{\phi})$$

converges weakly to normal distribution with covariance Σ_1.

PROOF. Let $U^n = \sqrt{n}(\phi(n) - \bar{\phi})$. As we have done in the proof of Theorem 1 while constructing $\phi^n(\cdot)$ from the sequence $\phi(n)$, let $U_n(\cdot)$ be a piecewise constant right continuous interpolation on $[0, \infty)$. So, $U_n(t) = U^{m(t+t_n)}$.

Let $\epsilon_n = \frac{1}{n+1}$, so $\phi(n)$ satisfies the equation

$$\begin{aligned}\phi(n+1) &= \phi(n) + \epsilon_n g(\phi(n)) + \epsilon_n d_n \qquad (8) \\ &= \phi(n) + \epsilon_n Y_n \qquad (9)\end{aligned}$$

where $g(\phi(n)) = C(\phi) - \phi$, and d_n is a Martingale difference. So, $\phi(n)$ satisfies Algorithm 10.2.1 in [18]. We used Theorem 10.2.1 in [18] for our proof, so let us check the assumptions of this theorem; A2.0 to A2.7.

We have already showed in the proof of Theorem 1, $g(\phi(n)) + d_n = I - \phi(n)$ and we know that $||g(\phi(n))||_2 < 1$ so $||M_n||_2 < 2$. Thus, $||Y_n|| = ||g(\phi(n)) + d_n|| \leq 2$. As a result, $\{Y_n I_{|\phi(n) - \bar{\phi}| \leq \phi}\}$ is uniformly integrable, which satisfies A.2.1.

As already shown, $\phi(n) \to \bar{\phi}$ so, A.2.2 holds.

The tightness condition, A.2.3, can be concluded from the theorem 10.4.1 in [18]. Also from the fact that $g(\phi(n)) = C(\phi) - \phi = 0$, assumption A.2.5 is satisfied.

By the assumption of C being smooth, $g(\phi(n)) = C(\phi) - \phi$ is also smooth, so assumptions A.2.4 and A.2.6 hold, where in our case $A = DC(\phi) - I$ is a Hurwitz matrix.

Define $\Sigma = diag\{C(\bar{\phi})\} - C^T(\bar{\phi})C(\bar{\phi})$. From $d_n = I^n - C(\phi(n))$ and $||M_n||_2 < 2$, it can be shown

$$E_n[(d_n)(d_n)^{'} I(\phi(n) - \bar{\phi}) \leq \rho] \to \Sigma$$

Thus, A.2.7 holds. Now, we can apply Theorem 10.2.1 in [18]. So, $U_n(\cdot)$ converges weakly in $D^m[0, \infty]$ to a stationary process $U(\cdot)$, where $U(\cdot)$ is the solution of:

$$U(t) = U(0) + \int_0^t (A + I/2)U(s)ds + W(t)$$

where $W(\cdot)$ is a Weiner process with covariance matrix Σ. We know that the solution of this SDE is Ornstein-Uhlenbeck Process and its stationary distribution is Gaussian with covariance matrix $\Sigma_1 = (A + I/2)^{-1}\Sigma$. $\square$

In the proof of Theorem 3, we use the standard analysis on the rate of convergence problem for general unconstrained stochastic approximations. Here, we show that

$$U^n = \sqrt{n}(\phi(n) - \bar{\phi})$$

converges weakly to a Gaussian distribution. So for large enough n, with high probability,

$$||\phi(n) - \bar{\phi}|| \leq \frac{\epsilon}{\sqrt{n}}$$

In other words, for sufficiently large number of people in the market, with high probability, the difference between the market share and the equilibrium point is less than $\frac{\epsilon}{\sqrt{n}}$ where ϵ is a function of J. In fact, Theorem 3 shows that the variance of the stationary measure is a function of the social influence.

Example.

Suppose there are $m = 3$ products in the market and their fitness' are $f_1 = 1$, $f_2 = 2$, and $f_3 = 3$.
When $J = 4$, there are 3 potential fixed points:

$$\begin{aligned} CR = \{&(0.9615, 0.0249, 0.0135),\\ &(0.0100, 0.8803, 0.1097),\\ &(0.0026, 0.0071, 0.9903)\} \end{aligned}$$

Running Monte Carlo simulation showes that the process converges to the first equilibrium with a probability close to .68 and converges to the second with probabilty close to .3.

When $J = 1$, there is only one fixed point:

$$CR = \{(0.0862, 0.2380, 0.675)\}$$

3. A SPECIAL CASE: LOGIT MODEL

Next, we will focus on the case when the choice probability function is logit. This is a special case of our model:

$$i \in \underset{j}{\operatorname{argmax}}(Jh(\phi_j(n)) + f_j + \epsilon_j)$$

when the function h is the identity function and the noise terms, ϵ_j, are independent and identically Gumbel distributed.

By finding an approximate solution for the equilibrium of the multinomial logit model, we have the following three observations: In case of weak social influence, i.e. when J is small enough, there is a unique equilibrium. On the other hand, with strong social influence, monopoly occurs. In this case, eventually any of the products can get the largest market share, so the number of equilibria is m. Another observation for the logit model is that inequality coefficient G increases with the social influence. These results support the experimental result of Salganik et.al. [27].

THEOREM 4. *For the logit choice function, when J is small enough, there is a unique equilibrium:*

$$\phi_i \approx \frac{e^{f_i}}{A + \sqrt{JB}} + \frac{Je^{2f_i}}{(A + \sqrt{BJ})^2}$$

where A and B are constants. When J is large enough, for each equilibrium, there exits some i such that $1 - \bar{\phi}_i = \Theta(e^{-J})$. *In other words, product i has monopoly in the market. Also, the inequality in the market, which is represented by the Gini coefficient,*

$$G = \frac{1}{2M} \sum_{\alpha,\beta} |\bar{\phi}_\alpha - \bar{\phi}_\beta|$$

has an increasing drift with J for the logit case.

According to Theorem 2, equilibrium points satisfy:

$$\begin{cases} C_i(\bar{\phi}) = \frac{e^{J\bar{\phi}_i + f_i}}{\sum_{i=0}^m e^{J\bar{\phi}_i + f_i}} = \bar{\phi}_i \\ DC(\bar{\phi}) - I \succeq 0 \end{cases} \tag{10}$$

Let $F(x) = x - \ln x$. When we define $u = \ln(\sum_{i=0}^m e^{J\bar{\phi}_i + f_i})$, from (10):

$$F(J\bar{\phi}_i) = u - f_i - \ln J$$

We can find two inverse functions for F. $F_1^{-1}(x)$ for $x \geq 1$ and $F_2^{-1}(x)$ for $x \leq 1$, where $F_1^{-1}(x)$ is an increasing function while $F_2^{-1}(x)$ is a decreasing function. Let $S = \{i : \bar{\phi}_i \leq \frac{1}{J}\}$. Therefore, equation (10) is equivalent to finding u such that:

$$\sum_{i \in S} \frac{1}{J} F_2^{-1}(u - f_i - \ln J) + \sum_{i \notin S} \frac{1}{J} F_1^{-1}(u - f_i - \ln J) = 1 \tag{11}$$

From the fact that $DC(\bar{\phi}) - I$ is positive semidefinite, we can conclude $|S| \geq m - 1$, i.e. there exists at most one i such that $\bar{\phi}_i \geq \frac{1}{J}$.

Let $\bar{f} = \max_i f_i$ and

$$J^* = \sum_i F_2^{-1}(\bar{f} - f_i + 1)$$

Equation (11) has a unique solution when $J \leq J^*$ if and only if $S = \{1, \cdots, m\}$. We refer to this case as weak social influence case.

Similarly, we can define some constant J_i such that if $J > J_i$ the equation (11) has a solution when $i \notin S$. We show that in this case, in equilibrium, ϕ_i is close to one and all other ϕ_j are close to zero. We call this case as strong social influence case.

In both of these cases, we can use the lemma below to find an approximation of the equilibrium points:

LEMMA 1.

$$x + \ln x + \frac{1}{2} \geq F_1^{-1}(x) \geq x + \ln x$$

$$e^{-x} e^{e^{-x+1}} \geq F_2^{-1}(x) \geq e^{-x} e^{e^{-x}}$$

By applying Lemma 1, we have the following bound for weak social influence case:

$$\phi_i \leq \frac{e^{f_i}}{.8A + \sqrt{\frac{1}{2}JB}} + \frac{Jce^{2f_i}}{(.8A + \sqrt{\frac{1}{2}BJ})^2} \tag{12}$$

$$\phi_i \geq \frac{e^{f_i}}{A + \sqrt{cJB}} + \frac{Je^{2f_i}}{(A + \sqrt{BJ})^2} \tag{13}$$

In the strong social influence case, using Lemma 1, we can find a bound for ϕ_i when $i \notin S$:

$$A_i e^{-J+\alpha} + cB_i J e^{-2(J-\alpha)} \geq 1 - \phi_i \geq A_i e^{-J} + B_i J e^{-2J}$$

By using the approximations derived above, we can observe that there exist increasing functions $U(J)$, $L(J)$ that bound the Gini coefficient:

$$U(J) \geq G(J) \geq L(J)$$

Now we look at the two cases above more specifically.

When $\phi_i \leq \frac{1}{J}$ for all i,

$$U(J) - L(J) \leq C_1 \max(J, C_2)$$

where C_1 and C_2 are constants.

When $\phi_i \geq \frac{1}{J}$ for some i, $U(J) - L(J) \leq Ce^{-J}$, where C is a constant.

4. CONCLUSION

We have presented an analysis for the dynamics of market share when social influence is present. By using techniques from stochastic approximation theory, which relate the limit behavior of a stochastic process to the limit behavior of a differential equation, we show that market share converges to an equilibrium in our model. We also focus on the case when the choice model is the multinomial logit model. In this special case, we show that inequality in the market increases with social influence, and with large enough social influence, monopoly occurs. Our observations for the logit model supports the empirical results of a study from a recent Web-based music market experiment [27].

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