

# Competitive Routing over Time

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## ABSTRACT

Congestion games are a fundamental and widely studied model for selfish allocation problems like routing and load balancing. An intrinsic property of these games is that players allocate resources simultaneously and instantly. This is particularly unrealistic for many network routing scenarios, which are one of the prominent application scenarios of congestion games. In many networks load travels along routes over time and allocation of edges happens sequentially. In this paper we consider two frameworks that enhance network congestion games with a notion of time. We propose *temporal network congestion games* that use coordination mechanisms – local policies that allow to sequentialize traffic on the edges. In addition, we consider *congestion games with time-dependent costs*, in which travelling times are fixed but quality of service of transmission varies with load over time. We study existence and complexity properties of pure Nash equilibria and best-response strategies in both frameworks. In some cases our results can be used to characterize convergence for various distributed dynamics.

## 1. INTRODUCTION

In this paper, we consider *network congestion games*, an intuitive game-theoretic model for competitive resource usage, which has attracted a great deal of attention recently [1, 11, 19]. They are central in modelling routing and scheduling tasks with distributed control [20]. Network congestion games can be described by a routing network and a set of

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players who each have a source and a target node in the network and choose a path connecting these two nodes. The quality of a player's choice is evaluated in terms of the total delay or latency of the chosen path. For this, every edge  $e$  has a latency function that increases with the number of players whose paths include edge  $e$ . This construction implicitly assumes that players use all edges on their paths instantaneously and simultaneously.

Depending on the application, it is, however, not always reasonable to assume that a player instantaneously allocates all edges on his chosen path. Consider for instance a road traffic network, in which players route cars to their destinations. Clearly, a traffic jam that delays people at rush hour might be harmless to a long distance traveller who reaches the same street hours later. In this case, it is more natural to assume that edges are allocated consecutively, and players take some time to pass an edge before they reach the next edge on their path.

In this paper, we study two different models that extend the standard model of network congestion games by a temporal component. We incorporate the assumption that on each edge the traffic over the edge must be sequentialized. This results in a scheduling problem with release dates on each edge, and our model uses the idea of *coordination mechanisms* [4, 6, 8, 14] that have been introduced in the context of machine scheduling and selfish load balancing [22]. In selfish load balancing each player has a task and has to assign it to one of several machines in order to minimize his completion time. A coordination mechanism is a scheduling algorithm run locally on each machine. Given an assignment of tasks to machines, the coordination mechanism run on a machine  $e$  gets as input the set of tasks assigned to  $e$  and their processing times on  $e$ . Based on this information it has to output a preemptive or non-preemptive schedule of the tasks on  $e$ . The coordination mechanism does not have access to any global information, like, e.g., the set of all tasks and their current allocation.

In *temporal congestion games*, which are studied in Section 3, we assume that each edge in a network congestion game is a machine equipped with a coordination mechanism, and each player has a task and chooses a path. Starting from their source, tasks travel along their path from one edge to another until they reach the target. They become available on the next edge of their path only after they have been processed completely on the previous edges. The player in-

curs as latency the total travel time that his task needs to reach the target. Each player then strives to pick a path that minimizes his travel time.

In our second model, which we term *congestion games with time-dependent costs* and study in Section 4, we assume that the travel time along each edge is a constant independent of the number of players using that edge. This model captures the property that increased traffic yields decreased quality of service for transmitting packets. We model this via a time-dependent cost function. We assume time is discretized into units (e.g., seconds), and the cost of an edge during a second depends on the number of players currently travelling on the edge. Each player now strives to pick a path that minimizes the total time-dependent costs during the travel time along the edges.

Our games extend atomic congestion games, which were initially considered by Rosenthal [19]. They are a vivid research area in (algorithmic) game theory and have attracted much research interest, especially over the last decade. A variety of issues have been addressed, most prominently complexity of computing equilibria [1, 11, 19] and bounding their inefficiency [3, 7, 21]. For an overview and introduction to the topic we refer to the recent expositions by Roughgarden [20] and Vöcking [22]. Addressing the notion of time in congestion games has only been started very recently in a number of papers [2, 12, 16]. Koch and Skutella [16] present a general model for flows over time using queueing models. Similarly, Anshelevich and Ukkusuri [2] derive a number of related results for similar model of flows over time. In contrast to our work both papers address non-atomic congestion games, in which players are infinitesimally small flow particles. Farzad et al. [12] consider a priority-based scheme for both, non-atomic and atomic games. In their model players have priorities, and a resource yields different latencies depending on the priority of a player allocating it. This includes an approach of Harks et al. [13] as a special case. While there can be different latencies for different players, this model does not include a more realistic “dynamic” effect that players delay other players only for a certain period of time. This is the case in our paper, as well as in [2, 16] for the non-atomic case.

## 1.1 Our Contribution

For temporal congestion games, we study four different (classes of) coordination mechanisms: (1) *FIFO*, in which tasks are processed non-preemptively in order of arrival. (2) *Non-preemptive global ranking*, in which there is a global ranking among the tasks that determines in which order tasks are processed non-preemptively (e.g., Shortest-First or Longest-First). (3) *Preemptive global ranking*, in which there is a global ranking that determines in which order tasks are processed and higher ranked tasks can preempt lower ranked tasks. (4) *Fair time-sharing*, in which all tasks currently located at an edge get processed simultaneously and each of them gets the same share of processing time.

For the FIFO policy (with unweighted and symmetric players) and the Shortest-First policy (with weighted but symmetric players) we show an interesting contrast of positive and negative results: even though computing a best response is NP-hard, there always exists an equilibrium, which can be computed in polynomial time. Moreover, the equilibrium is not only efficiently computable, but we present natural dynamics in which uncoordinated agents are able to find

an equilibrium quickly even without solving computationally hard problems. We then show that Shortest-First is the only global ranking that guarantees the existence of Nash equilibria in the non-preemptive setting. That is, for any other global ranking (e.g., Longest-First) there exist temporal congestion games without equilibria. In contrast to this, we show that preemptive games are potential games for every global ranking and that uncoordinated agents reach an equilibrium quickly. Finally, we show that even though fair time-sharing sounds like an appealing coordination mechanism it does not guarantee the existence of equilibria, not even for unweighted and symmetric players.

For the second model, congestion games with time-dependent costs, we prove that these games can be reduced to standard congestion games. Hence, they are potential games and also the known results on the price of anarchy carry over. We prove that computing a best response in these games in NP-hard in general and that already for a very restricted class with polynomially bounded delays and acyclic networks computing an equilibrium is PLS-complete.

All missing proofs are presented in the Appendix.

## 2. NOTATION

A *network congestion game* is described by a directed graph  $G = (V, E)$ , a set  $\mathcal{N} = \{1, \dots, n\}$  of *players* with *source nodes*  $s_1, \dots, s_n \in V$  and *target nodes*  $t_1, \dots, t_n \in V$ , and a non-decreasing *latency function*  $\ell_e: [n] \rightarrow \mathbb{R}_{\geq 0}$  for each edge  $e$ . We will only consider linear latency functions of the form  $\ell_e(x) = a_e x$  in this paper. For such functions, we call  $a_e$  the *speed* of edge  $e$ . The *strategy space*  $\Sigma_i$  of a player  $i \in \mathcal{N}$  is the set of all simple paths in  $G$  from  $s_i$  to  $t_i$ . Given a *state*  $P = (P_1, \dots, P_n) \in \Sigma = \Sigma_1 \times \dots \times \Sigma_n$  of a network congestion game, we denote by  $n_e(P) = |\{i \in \mathcal{N} \mid e \in P_i\}|$  the *congestion* of edge  $e \in E$ . The *individual latency* that a player  $i$  incurs is  $\ell_i(P) = \sum_{e \in P_i} \ell_e(n_e(P))$ , and every player is interested in choosing a path of minimum individual latency. We call a network congestion game *weighted* if additionally every player  $i$  has a weight  $w_i \geq 1$ . Then the congestion on a edge  $e$  is defined as  $n_e(P) = \sum_{i: e \in P_i} w_i$  and the domain of the latency functions is extended appropriately. We call a congestion game *symmetric* if every player has the same source node and every player has the same target node. If not explicitly mentioned otherwise, we consider unweighted and asymmetric congestion games.

We incorporate time into the standard model in two different ways. The introduction of time alters the individual latency functions  $\ell_i$ . The specific definition will be given in the sections below. For our altered games we are interested in stable states, which are pure strategy Nash equilibria of the games. Such an equilibrium is given by the condition that each player plays a best response and has no unilateral incentive to deviate, i.e.,  $P$  is a pure *Nash equilibrium* if for ever player  $i$  and every state  $Q$  that is obtained from  $P$  by replacing  $i$ 's path by some other path, it holds  $\ell_i(P) \leq \ell_i(Q)$ , where  $\ell_i$  denotes the (altered) latency function of player  $i$ . We will not consider mixed Nash equilibria in this paper, and the term Nash equilibrium refers to the pure version throughout.

## 3. COORDINATION MECHANISMS

In this section we consider *temporal network congestion games*. These games are described by the same parameters

as standard weighted network congestion games with linear latency functions. However, instead of assuming that a player allocates all edges on his chosen path instantaneously, we consider a scenario in which players consecutively allocate the edges on their paths. We assume that each player has a weighted task that needs to be processed by the edges on his chosen path.

Formally, at each point in time  $\tau \in \mathbb{R}_{\geq 0}$ , every task  $i$  is located at one edge  $e_i(\tau)$  of its chosen path, and a certain fraction  $f_i(\tau) \in [0, 1]$  of it is yet unprocessed on that edge. The coordination mechanism run on edge  $e$  has to decide in each moment of time which task to process. If it decides to work on transmitting task  $i$  for  $\Delta\tau$  time units starting at time  $\tau$ , then the unprocessed fraction  $f_i(\tau + \Delta t)$  of task  $i$  at time  $\tau + \Delta t$  is  $\max(0, f_i(\tau) - \Delta\tau/(a_e w_i))$ . In total, task  $i$  needs  $a_e w_i$  time units to finish on edge  $e$ . Once  $f_i(\tau) = 0$ , task  $i$  arrives at the next edge on its path and becomes available for processing. The coordination mechanism can base the decision on which task to process next for how long only on local information available at the edge — such as the weights and arrival times of those tasks that have already arrived at the edge. The individual latency  $\ell_i(P)$  of player  $i$  in state  $P$  is the time at which task  $i$  is completely finished on the last edge of  $P_i$ .

In the following we examine several natural scheduling policies for the edges with respect to the existence and complexity of Nash equilibria.

### 3.1 The FIFO Policy

One of the most natural coordination mechanisms is the FIFO policy. If several tasks are currently located at the same edge, then the one that has arrived first is executed non-preemptively until it finishes. In the case of ties, there may be an arbitrary tie-breaking that is consistent among the edges. First we consider symmetric games with unweighted players and prove among other results that these games always possess Nash equilibria. After that we show that for the generalizations to either asymmetric games or games with weighted players, the existence of equilibria is not guaranteed anymore.

At first we treat symmetric temporal network congestion games with unweighted players. For these games we obtain an interesting contrast of positive and negative results: even though computing a best response is NP-hard, there always exists a Nash equilibrium, which can be computed in polynomial time. Moreover, the equilibrium is not only efficiently computable, but uncoordinated agents are able to find it quickly even without solving computationally hard problems.

**THEOREM 3.1.** *For symmetric temporal network congestion games with unweighted players, Nash equilibria always exist if tasks are forwarded according to the FIFO policy. Moreover, a Nash equilibrium can be computed efficiently.*

**PROOF.** Let us assume without loss of generality that players are numbered according to their rank in tie-breaking, i.e. 1 is the highest ranked player, and  $n$  is the lowest ranked player. Assume that we start in an arbitrary state of the game in which the players have chosen arbitrary paths. Below we define a subclass of best responses, which we call *greedy best responses*. We claim that we obtain an equilibrium if we let the players  $1, 2, \dots, n$  play each one greedy best response in this order. To prove this, assume that the

players  $1, \dots, i$  are already playing greedy best responses, and now let player  $i + 1$  also change his strategy to a greedy best response. We show that after this strategy change the players  $1, \dots, i$  are still playing greedy best responses, which proves by induction that a Nash equilibrium is reached once every player has played a greedy best response. We prove the following invariant: if the players  $1, \dots, i$  play greedy best responses, then none of them can be delayed at any node by a lower ranked player  $j > i$ . Furthermore, the current paths of the players  $1, \dots, i$  are (greedy) best responses no matter which paths the other players  $j > i$  choose. For the first player, every best response is defined to be a greedy best response. Given this definition, we argue that the aforementioned claim is true for  $i = 1$ : We consider the network  $G = (V, E)$  as a weighted graph in which every edge  $e \in E$  has weight  $a_e$ . Let  $P_1$  denote a shortest path in this weighted graph from the source  $s$  to the target  $t$  and let  $a^*$  denote its length. If the highest ranked player chooses path  $P_1$ , then it cannot be delayed at any node  $v$  by any other player  $j$ , as otherwise,  $j$  would have found a shorter path from  $s$  to  $v$ , contradicting the choice of  $P_1$  as shortest path from  $s$  to  $t$ . Hence, when player 1 chooses path  $P_1$  its total latency is  $a^*$  no matter which paths the other players choose. Clearly, the length  $a^*$  is also a lower bound on the time it takes any player to reach the target, and hence, choosing  $P_1$  is a (greedy) best response for player 1. Moreover, any (greedy) best response of player 1 corresponds to a shortest path  $P_1$  in the aforementioned weighted graph. Now let us recursively define what a greedy best response is for player  $i + 1 > 1$ . For this, assume that the players  $1, \dots, i$  play already greedy best responses. Based on the paths chosen by these players, we construct a distance function  $d: V \rightarrow \mathbb{R}_{\geq 0}$  for the network  $G = (V, E)$ , which eventually tells us for every node how long it takes player  $i + 1$  to get there. The construction of this distance function follows roughly Dijkstra's algorithm: Let  $I \subseteq V$  denote the set of nodes that have already an assigned distance. We start with  $I = \{s\}$  and  $d(s) = 0$ . For extending the set  $I$ , we crucially use the fact that the players  $1, \dots, i$  cannot be delayed by other players, which means that every edge  $e \in E$  has a fixed schedule saying when it is used by the players  $1, \dots, i$  and when it is available for player  $i + 1$ . These fixed schedules imply in particular that for every node  $v \in V$  there exists a shortest path  $s, v_1, \dots, v_k = v$  for player  $i + 1$  from  $s$  to  $v$  such that every subpath  $s, v_1, \dots, v_{k'}$  is a shortest path from  $s$  to  $v_{k'}$ . Hence, taking into account the fixed schedules and the possible delays that they induce on player  $i + 1$ , we can extend the set  $I$  as in Dijkstra's algorithm, that is, we insert the node  $v \in V \setminus I$  into  $I$  that minimizes  $\min_{u \in I} d(u) + \ell(u, v)$ , where  $\ell(u, v)$  denotes the time it takes player  $i + 1$  to get from  $u$  to  $v$  if he arrives at node  $u$  at time  $d(u)$ . The distance  $d(v)$  assigned to node  $v$  is  $\min_{u \in I} d(u) + \ell(u, v)$ . This algorithm constructs implicitly a path from  $s$  to any other node. Any path from  $s$  to  $t$  that can be constructed by this algorithm (the degree of freedom is the tie-breaking) is called a greedy best response of player  $i + 1$ .

It is easy to see that any such greedy best response is really a best response for player  $i + 1$  if only the players  $1, \dots, i + 1$  are present as there is no quicker way to reach the target  $t$  from the source  $s$  given the current paths of the players  $1, \dots, i$ . To complete the proof we need to argue that player  $i + 1$  cannot be delayed by players  $j > i + 1$ . Assume there is a node  $v$  and a player  $j > i + 1$  such that

$j$  arrives earlier at node  $v$  than  $i + 1$ . This contradicts the construction of the path as it implies that there is a faster way to get from the source  $s$  to the node  $v$ . Again this uses crucially that the players  $1, \dots, i$  cannot be delayed by lower ranked players.  $\square$

In the proof we define the notion of greedy best responses. Basically, a path  $s, v_1, \dots, v_k, t$  is a greedy best response for player  $i$  if any subpath  $s, v_1, \dots, v_{k'}$  is a shortest path from  $s$  to  $v_{k'}$ . Note that this is not the case in arbitrary best responses: it could, for example, be the case that player  $i$  has to wait at some node  $v_{k'}$  because it is blocked by a player with a higher rank. Then, the subpath from  $s$  to  $v_{k'}$  is not necessarily the shortest possible path in every best response. However, we believe that the restriction to greedy best responses is a natural assumption on the players' behavior.

The proof of the previous result implies not only that a Nash equilibrium always exists, but it also shows that players reach it in a distributed fashion using different forms of dynamics. Consider the following Nash dynamics among the players. At each point in time, one player is picked and allowed to change his strategy. We show below that in general it is NP-hard for this player to decide whether it can decrease his latency by changing his path. In that case, the player might stick to his current path or make an arbitrary strategy change, following some heuristic. However, at each point in time there is one player who can easily find a (greedy) best response, namely the highest ranked player  $i + 1$  that does not play a greedy best response, but the players  $1, \dots, i$  do. We assume that this player changes to a greedy best response when he becomes activated. We also assume that a player who is already playing a greedy best response does not change his strategy when he becomes activated. A *round* is a sequence of activations in which every player gets at least once the chance to change his strategy. From the proof of Theorem 3.1 it follows easily that a Nash equilibrium is reached after at most  $n$  rounds. We are interested in particular in the *random greedy best response dynamics*, in which in each iteration the activated player is picked uniformly at random, and the *concurrent best response dynamics*, in which in each iteration all players are simultaneously allowed to change their strategy, each one with some constant probability  $0 < p_i \leq 1$ . In both these dynamics, rounds are polynomially long with high probability. Summarizing, we obtain the following corollary.

**COROLLARY 3.2.** *In every symmetric temporal network congestion game with unweighted players and FIFO policy, it takes at most  $n$  rounds to reach a Nash equilibrium. In particular, the random and concurrent greedy best response dynamics reach a Nash equilibrium in expected polynomial time.*

Finally, we turn to the hardness result.

**THEOREM 3.3.** *Computing best responses is NP-hard in symmetric temporal network congestion games with unweighted players if tasks are forwarded according to the FIFO policy.*

Any relaxation of the restrictions to unweighted players or asymmetric games leads to possible non-existence of Nash equilibria.

**THEOREM 3.4.** *There exist temporal congestion games using the FIFO policy that do not possess Nash equilibria and (1) are symmetric with weighted players, or (2) are asymmetric with unweighted players.*

## 3.2 Non-preemptive Global Ranking

Another natural approach is to assume that there is a global ranking  $\pi: [n] \rightarrow [n]$  on the set of tasks with  $\pi(1)$  being the task with the highest priority and so on. In this case, tasks are scheduled non-preemptively according to this ranking. When an edge  $e$  becomes available, the highest ranked task  $i$  that is currently located at the edge is processed non-preemptively. That means it exclusively uses  $e$  for  $a_e w_i$  time units. After that, task  $i$  moves to the next edge on its path, and  $e$  selects the next task if possible. In this section, we consider mainly weighted games and assume without loss of generality that  $w_1 \leq w_2 \leq \dots \leq w_n$ .

We first show that all results that we obtained for the FIFO policy can be transferred to the Shortest-First policy. Then we show that this is the only global ranking that guarantees the existence of equilibria.

At first we consider the identity ranking  $\pi(i) = i$ , which corresponds to the *Shortest-First policy*. It is easy to see that Theorem 3.1 and Corollary 3.2 carry over to this case. The proof for FIFO was essentially based on the observation that once all players  $1, \dots, i$  play a (greedy) best response, they cannot be affected by the lower ranked players. This is even more true for the Shortest-First policy as the lower ranked players have now the additional disadvantage of having a longer processing time.

**THEOREM 3.5.** *For symmetric weighted temporal network congestion games with Shortest-First policy Nash equilibria always exist. Moreover, a Nash equilibrium can be computed efficiently. In such games it takes at most  $n$  rounds to reach a Nash equilibrium. In particular, the random and concurrent greedy best response dynamics reach a Nash equilibrium in expected polynomial time.*

Also the hardness result in Theorem 3.3 carries over easily. We just need to set all weights to 1 and embed the same tie-breaking as in the proof of Theorem 3.3 in the ranking  $\pi$ . In that construction only the tie-breaking is important; the FIFO policy is never used, that is, it never happens that at a busy edge two players arrive one after another.

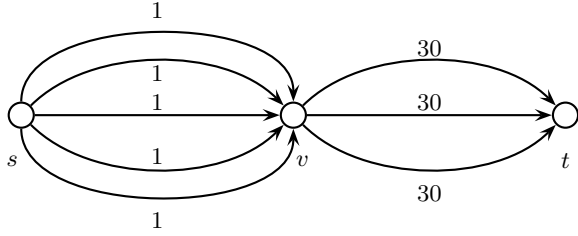
**THEOREM 3.6.** *In (unweighted) symmetric temporal network congestion games with Shortest-First policy computing a best response is NP-hard.*

Although the previous arguments guarantee existence and convergence to a Nash equilibrium for the Shortest-First policy, such games are not necessarily potential games.

**PROPOSITION 3.7.** *There is a symmetric temporal network congestion game with Shortest-First policy that is no potential game.*

In the case of more general rankings we show that for any global ranking other than Shortest-First, a symmetric temporal network congestion game without Nash equilibrium exists.

**THEOREM 3.8.** *For any given set of player task weights  $w_1 \leq \dots \leq w_n$  and any ranking  $\pi$  other than the identity,*



**Figure 1:** A game without a Nash equilibrium for 5 tasks. The ranking is  $(1, 2, 4, 3, 5)$ , and we assume  $w_3 < w_4 < 31w_3$ . Task 4 always strives to join task 3 on an edge in the first layer. Then it can be processed first on one of the edges in the second layer. Task 3 always strives to move away from any edge shared with task 4 in the first layer.

there exists a graph with latency functions such that the resulting symmetric temporal congestion game does not possess a Nash equilibrium.

**PROOF.** Let  $j$  denote the index with the property that for player  $i \in [j - 1]$  task  $w_i$  has the  $i$ -th highest priority, but task  $w_j$  does not have the  $j$ -th highest priority. Let  $w_k$  with  $k > j$  be the task with the  $j$ -th highest priority. The network we construct consists of two layers of parallel links. On the first layer there are  $n$  edges with speed 1. On the second level there are  $k - 1$  slow edges with speed  $a$ , where  $a$  is sufficiently large.

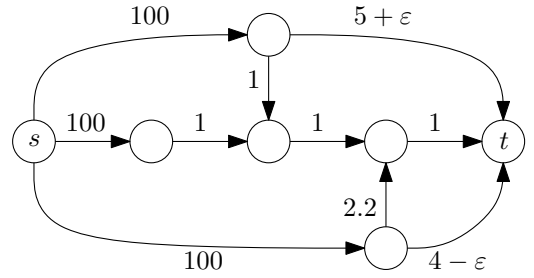
Now consider an arbitrary state of this game and assume that the players  $w_1, \dots, w_{j-1}$  have chosen disjoint paths, which must be true in every Nash equilibrium. If one of the players  $w_j, \dots, w_{k-1}$  has to share its edge on the first level with another player with a higher priority, then it will change to an unused edge on the first level. This edge is guaranteed to exist because there are  $n$  parallel links. On the other hand, if none of the players  $w_j, \dots, w_{k-1}$  shares its edge with another task of higher priority, then the tasks  $w_1, \dots, w_{k-1}$  are the first ones that arrive at the intermediate node. Hence, for  $a$  sufficiently large,  $w_k$  has to wait for a long time until it can pass the second level. Hence,  $w_k$  has an incentive to change to an edge on the first level that is used by a task  $w_l$  with  $l \in \{j, \dots, k - 1\}$ . Since  $w_k$  has a higher priority, it will be able to arrive before  $w_l$  and it does not have to wait at the intermediate node.  $\square$

The previous proof yields the same result for asymmetric games with the Shortest-First policy.

**COROLLARY 3.9.** *For any given set of player task weights  $w_1, \dots, w_n$  and the Shortest-First policy, there exist a graph and latency functions such that the resulting asymmetric temporal congestion game does not have a Nash equilibrium.*

### 3.3 Preemptive Global Ranking

When we assume a global ranking and allow preemptive execution, it is possible to adapt the arguments of Theorem 3.1 to asymmetric weighted games. Indeed, all arguments in this section work for a very general class of preemptive games with unrelated edges. That is, every player  $i$  has its own processing time  $p_{ie}$  for every edge  $e$ . These processing times may even depend on the time at which player  $i$  reaches edge  $e$ . The only assumption we need to make is



**Figure 2:** Symmetric game with unweighted players and without Nash equilibrium for Time-Sharing.

that the processing times are monotone in the sense that if task  $i$  reaches edge  $e$  at time  $t$ , then it does not finish later than when it reaches edge  $e$  at time  $t' > t$ .

**THEOREM 3.10.** *Every asymmetric temporal congestion game with preemptive policy  $\pi$  is a potential game. A Nash equilibrium exists and can be computed in polynomial time. For any state and any player, a best response can be computed in polynomial time.*

Similarly we can adapt the previous observations in Corollary 3.2 and show that various improvement dynamics converge in polynomial time.

**COROLLARY 3.11.** *In every asymmetric temporal network congestion game with any preemptive policy  $\pi$ , it takes at most  $n$  rounds to reach a Nash equilibrium. The expected number of iterations to reach a Nash equilibrium for random and concurrent best response dynamics is bounded by a polynomial.*

### 3.4 Fair Time-Sharing

In this section we consider fair time-sharing, a natural coordination mechanism based on the classical idea of fair queueing [17] and uniform processor sharing [15]. When multiple tasks are present at an edge  $e$ , they are all processed simultaneously, and each one of them gets the same share of bandwidth or processing time. As in generalized processor sharing [18] we assume round-robin processing with infinitesimal time slots. Even though such a fairness property is desirable, the following theorem shows that Nash equilibria are not even guaranteed to exist for symmetric games with unweighted players. This obviously remains true for extensions, where bandwidth is allocated using player priority weights (that might be different from the task weights), which are used e.g. in weighted fair queueing [9].

**THEOREM 3.12.** *For unweighted players and symmetric networks, Nash equilibria do not necessarily exist if tasks are forwarded according to the time-sharing policy.*

**PROOF.** The instance shown in Figure 2 has three players. As the three edges leaving the source  $s$  are very slow, in any Nash equilibrium all three players will use different edges leaving the source. We assume without loss of generality that the first player chooses the upper edge, the second player chooses the middle edge, and the third player chooses the lower edge. Then players 1 and 3 have still two alternatives how to continue, whereas the path of player 2 is already determined. The speeds of the edges are chosen such that

player 1 wants to use the edge with speed  $5 + \varepsilon$  if and only if player 3 does not use the edge with speed  $4 - \varepsilon$ . On the other hand, player 3 wants to use the edge with speed  $4 - \varepsilon$  if and only if player 1 uses the edge with speed  $5 + \varepsilon$ , which completes the proof.  $\square$

In contrast to this negative result, Dürr and Nguyen [10] show that time-sharing on parallel links is always a potential game, even for unrelated machines (edges). That is, for parallel links Nash equilibria always exist. The potential function used in Dürr and Nguyen can be rewritten as the sum of the completion times (individual latencies) of the player. It is known [5] that a schedule minimizing this sum can be computed in polynomial time. Such a global minimum of the potential function must obviously be a pure Nash equilibrium for the time-sharing policy, yielding the following corollary.

**COROLLARY 3.13.** *For parallel links and unrelated tasks, a Nash equilibrium can be computed efficiently for the time-sharing policy.*

#### 4. CONSTANT TRAVEL TIMES AND QUALITY OF SERVICE

In this section we consider *network congestion games with time-dependent costs*. Again, players consecutively allocate the edges on their paths. This time, however, the travel time along an edge  $e$  in the network is fixed to a constant delay  $d_e$ . If a player chooses a path along the edges  $e_1, e_2, \dots$ , then he arrives at  $e_2$  at time  $d_1$  and at  $e_3$  at time  $d_1 + d_2$ , and so on. This travel time through the network is independent of how many other players allocate any of the edges. In this section, we only consider asymmetric games. For the strategic part we assume that each edge generates a separate usage cost  $c_e$  per time unit. This could, for instance, measure the quality of service that is enjoyed by the players during transmission. The cost depends on the number of players allocating the edge at a given point in time. In particular, edge  $e$  has a cost function  $c_e: [n] \rightarrow \mathbb{N}$  that describes the cost for allocating it for one second in terms of the current number of players. If for a state  $P$  an edge  $e$  is shared at time  $\tau$  by  $n_e(\tau, P)$  players, all these players get charged cost  $c_e(n_e(\tau, P))$ . The cost incurred by player  $i$  on a path  $P_i = (e_1, \dots, e_l)$  is then

$$\ell_i(P) = \sum_{j=1}^l \sum_{\tau=\tau_j}^{\tau_j+d_{e_j}-1} c_{e_j}(n_{e_j}(\tau, P)),$$

where  $\tau_1 = 0$  and  $\tau_j = \sum_{k=1}^{j-1} d_{e_k}$ .

It turns out that this model is equivalent to a regular congestion game. For each edge and each time unit we introduce a resource  $r_{e,\tau}$  and modify the strategy spaces as follows: For a path  $P = (e_1, \dots, e_l)$  the new strategy includes all resources  $r_{e_j,\tau}$  for  $\tau = \tau_j, \dots, \tau_j + d_{e_j} - 1$  and  $j = 1, \dots, l$ . This is a regular congestion game with latencies given by the time costs. Results on the existence of Nash equilibria and the price of anarchy carry over.

**COROLLARY 4.1.** *Network congestion games with time-dependent costs are equivalent to a class of regular congestion games. In particular, there is a pure Nash equilibrium in every game, and any better-response dynamics converges.*

However, as the standard congestion game obtained by this reduction might have a large number of resources. It

is not necessarily a network congestion game, so complexity results do not carry over. In particular, computing a best response is NP-hard. Computing a Nash equilibrium is PLS-complete, even in a very restricted special case.

**THEOREM 4.2.** *Computing a best response in network congestion games with time-dependent costs is NP-hard. For polynomially bounded delays and acyclic networks, best responses can be computed efficiently, but computing a Nash equilibrium is PLS-complete.*

**PROOF.** The NP-hardness of computing a best response follows easily with a reduction from the PARTITION problem. The input to this problem consists of  $n$  integers  $w_1$  to  $w_n$ . One has to decide if there exists a subset of these numbers that add up to exactly  $W/2$ , where  $W = \sum_{i=1}^n w_i$ . We construct a graph with vertices  $s = v_0, v_1, \dots, v_n, t$  and consider the best response of a player whose source node is  $s$  and whose target node is  $t$ . Between each pair of nodes  $(v_i, v_{i+1})$  there are two parallel edges with delays  $w_{i+1}$  and 0, respectively, and costs 0. In addition to this, there is one edge  $e$  from  $v_n$  to  $t$  with delay 1 and cost function  $c_e(n_e) = n_e$ . We assume that the other players have chosen paths such that in the time interval from 0 to  $W + 1$  always one player is using edge  $e$ , except for the interval from  $W/2$  to  $W/2 + 1$ . That is, only if the player manages to arrive at node  $v_n$  exactly at time  $W/2$ , then his costs will be 1. Otherwise, they will be at least 2. Any path from  $s$  to  $v_n$  corresponds to a subset of the weights  $w_i$ , and hence, the player has a strategy with costs 1 if and only if the partition instance is solvable. This proves the NP-hardness.

Let us now turn to acyclic networks with polynomially bounded delays. For this restricted case best responses can be computed efficiently by standard dynamic programming: we store for each of the polynomially many points in time  $\tau$  and every node  $v$  the least expensive path that arrives at  $v$  exactly at time  $\tau$ . First, we fill this table, taking into account only paths of length at most 1. From this, we can easily compute another table taking into account paths of length at most 2, and so on. This approach uses the fact that the network is acyclic, and it proves that the problem of computing a Nash equilibrium belongs to PLS.

For the completeness, we use the reduction in [1] for asymmetric network congestion games. This reduction has the property that it generates only acyclic networks. We will argue that there is a generic way to transform a standard network congestion game with acyclic network  $G$  into an acyclic network congestion game with time-dependent costs and polynomially bounded delays. For this, we take the network  $G$  and compute a topological ordering of the nodes. Let us assume without loss of generality that this ordering is  $v_1, v_2, \dots, v_k$ , where  $v_1$  has no incoming and  $v_k$  has no outgoing edges. If the source node of a player is  $v_i$ , then we introduce a new source node  $s_i$  for that player, which is connected by an edge with delay  $i$  and costs 0 to node  $v_i$ . This allows us to choose polynomially bounded delays for all edges such that every player whose path includes a node  $v_i$  arrives at this node at exactly time  $i$ . Hence, if we can keep the cost functions, this congestion game with time-dependent costs behaves exactly as the standard congestion games as players are now synchronized.  $\square$

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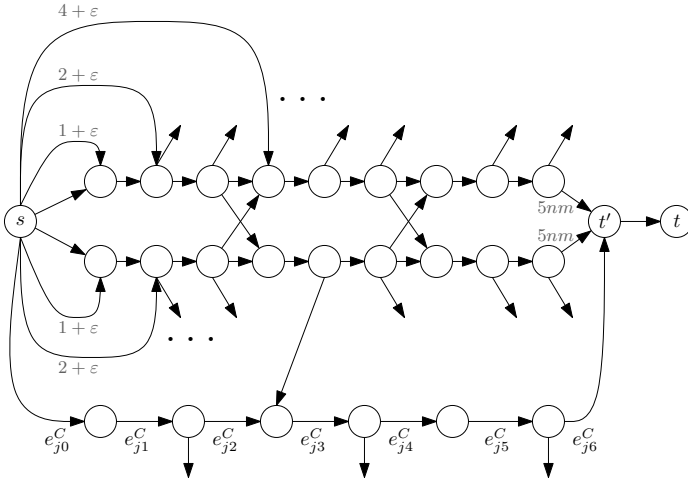
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## APPENDIX

### A. PROOF OF THEOREM 3.3

We show how to reduce instances of 3-SAT to symmetric temporal congestion games with unweighted players. Let an arbitrary instance for 3-SAT with variables  $x_1, \dots, x_n$  and clauses  $C_1, \dots, C_m$  be given, and assume that  $C_j = l_{j1} \vee l_{j2} \vee l_{j3}$ , where every literal  $l_{jk}$  is either  $x_i$  or  $\bar{x}_i$  for one  $i$ . We assume that the literals are ordered such that  $l_{j1}$  belongs to a variable  $x_i$  and  $l_{j2}$  belongs to a variable  $x_{i'}$  with  $i < i'$ . We assume the same monotonicity for  $l_{j2}$  and  $l_{j3}$ . The temporal congestion game that we construct has  $1 + m + 4nm$  players, one player  $p^D$ , who we call the decider and who is supposed to play a best response, one player  $p_j^C$  for every clause  $C_j$ , and four players  $p_{ij}^0, p_{ij}^1, \tilde{p}_{ij}^0, \tilde{p}_{ij}^1$  for every  $i \in [n]$  and  $j \in [m]$ . For the construction it is only important that all players have higher priorities than the decider  $p^D$ , and that the players  $p_{ij}^0$  and  $p_{ij}^1$  have higher priorities than the clause players.

Figure 3 depicts the network that we construct: there are two rows of nodes. Both rows are subdivided into  $n$  blocks of  $m + 1$  nodes each. At the end of a block, there is the possibility to switch from the upper to the lower row or vice versa. Intuitively, each block corresponds to one variable  $x_i$  and the decider either uses the upper row in the block, corresponding to  $x_i = 0$ , or he uses the lower row, corresponding to  $x_i = 1$ . Both rows lead to a vertex  $t'$  from which there is a direct edge to the target  $t$ . In addition to these edges, there is a direct edge from the source  $s$  to each node in the two rows, except for the nodes from which switching the rows is possible. All edges in the two rows (including the edges from  $s$  to the first nodes in the rows) have a speed of 1. The speed of the edges from the last nodes in the rows to  $t'$  is  $5nm$ . The speed of the edge from  $t'$  to  $t$  is 1. The speeds of the direct edges from  $s$  to the nodes in the two rows are  $1 + \varepsilon$ ,  $2 + \varepsilon$ , and so on, for a small  $\varepsilon > 0$ . That is, taking the direct edges is slightly slower than following a path along the rows (assuming no delays occur). Also all nodes in the two rows, except for the first nodes of every block, have an outgoing edge with speed 1, which we



**Figure 3: Construction in the proof of Theorem 3.6.** Gray labels indicate speeds, black labels are the names of the edges. In this example,  $n = 3$ ,  $m = 2$ , and the shown clause  $C_j$  has  $\bar{x}_2$  as second literal  $l_{j2}$ .

describe later in detail.

The current strategies of the players  $p_{ij}^0$  and  $p_{ij}^1$  are chosen as follows: player  $p_{ij}^1$  uses the direct edge from the source node  $s$  to the  $j$ -th node in the  $i$ -th block of the upper row, then he follows the edge in this row to the  $j+1$ -th node in the  $i$ -th block from which he leaves the component. Player  $p_{ij}^0$  does exactly the same but in the lower row. For each variable  $x_i$ , the decider can choose whether to delay all players  $p_{ij}^1$  or all players  $p_{ij}^0$  by choosing the upper or the lower row, respectively. In any case, the decider will reach node  $t'$  at time  $6nm$ .

In addition to the players described so far, we have the clause players  $p_j^C$ . Each of these players uses its own path with seven edges  $e_{j0}^C, \dots, e_{j6}^C$  from the source node  $s$  to the node  $t'$ . If a clause player is not delayed by any other player, then he reaches node  $t'$  at time  $6nm$ . As the clause players have a higher priority than the decider  $p^D$ , an undelayed clause player will delay the decider on its way from  $t'$  to  $t$ . If the decider reaches  $t'$  before all clause players he will have a total latency of  $6nm + 1$  for reaching  $t$ , otherwise his latency is at least  $6nm + 2$  as he has to wait at  $t'$ .

The decider can delay clause players by making the right choices between the upper and lower row. For this to work, we have to further specify the current paths of the players  $p_{ij}^0$  and  $p_{ij}^1$ . As we have described above, they use a direct edge to one node in the two rows, follow one edge in the row, and leave the row afterwards. The edges leaving the rows connect to the paths of the clause players as follows: Let us consider a clause player  $p_j^C$  for the clause  $C_j = l_{j1} \vee l_{j2} \vee l_{j3}$ . If  $l_{j1} = x_i$  for one  $i$ , then we choose one player  $p_{ik}^1$  that was not used in other clauses yet, and connect the outgoing edge by which he leaves the upper row to the starting point of the edge  $e_{j1}^C$ . The current path of player  $p_{ik}^1$  follows the edge  $e_{j1}^C$ . We assume that the endpoint of  $e_{j1}^C$  is connected by a very slow edge to the target  $t$ . The path of  $p_{ik}^1$  is terminated by using this edge. If  $l_{j1} = \bar{x}_i$ , then we analogously pick a player  $p_{ik}^0$  and finish his path in the same way. We do analogous constructions for the literals  $l_{j2}$  and  $l_{j3}$  and the

edges  $e_{j3}^C$  and  $e_{j5}^C$ . The speeds of the edges  $e_{j1}^C$ ,  $e_{j3}^C$ , and  $e_{j5}^C$  are 1. The speeds of the other edges on  $p_j^C$ 's path are chosen such that the clause player  $p_j^C$ , if not delayed, reaches the starting nodes of  $e_{j1}^C$ ,  $e_{j3}^C$ , and  $e_{j5}^C$  exactly at the same time as the corresponding players  $p_{ik}^0$  or  $p_{ik}^1$  and such that he reaches  $t'$  exactly at time  $5nm$ . Such speeds exist as every player  $p_{ik}^0$  or  $p_{ik}^1$  reaches the path of the clause player at time  $nm$  at the latest and the literals in every clause satisfy the aforementioned monotonicity condition. As we have  $m$  players  $p_{ik}^0$  and  $p_{ik}^1$  for every  $i$ , we can find a unique such player for every clause in which the literal  $x_i$  or  $\bar{x}_i$  occurs. For players  $p_{ik}^0$  and  $p_{ik}^1$  that are not needed, we assume that the corresponding outgoing edge leaving the rows is a very slow connection to the target  $t$ .

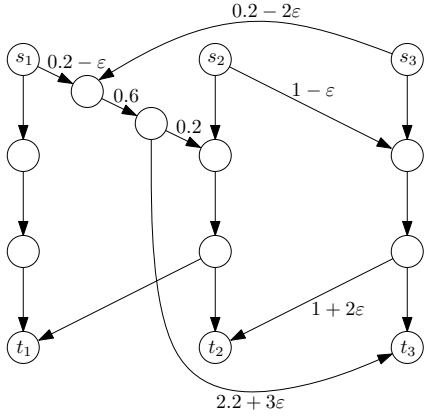
As the players  $p_{ik}^0$  and  $p_{ik}^1$  have higher priorities than the clause players, they will delay the clause players if they are not delayed by the decider. For each variable the decider can choose whether to delay all players  $p_{ik}^0$ , which corresponds to setting  $x_i = 0$ , or to delay all players  $p_{ik}^1$ , which corresponds to  $x_i = 1$ . The goal of the decider is to choose a path such that for every clause  $C_j$  at least one of the corresponding players  $p_{ik}^0$  or  $p_{ik}^1$  is not delayed. If the SAT formula is satisfiable, then he can achieve this and reach the target  $t$  in time  $6nm + 1$ .

To complete the reduction we show that the decider cannot reach the target in time  $6nm + 1$  if the formula is not satisfiable. If the decider sticks to the edges in the rows, then he cannot achieve this goal as he will be delayed at node  $t'$  by at least one clause player. He also cannot use the direct edges from the source  $s$  to the nodes in the rows as this will slow him down by  $\varepsilon$  (actually by more than  $\varepsilon$  as those edges are used by players with higher priority). So the only possibility left for the decider is to follow the edges in the rows up to some node and to use the edge to one of the clause players' paths from there. This might indeed be beneficial as the decider reaches a node on the path before the clause player. To prevent this from happening, we introduce the players  $\tilde{p}_{ij}^0$  and  $\tilde{p}_{ij}^1$ . These players use paths that are completely disjoint from the paths of the other players, except for a single edge. Each player  $\tilde{p}_{ij}^0$  uses the edge that connects  $p_{ij}^0$ 's path from the rows to the path of the clause player. The path of  $\tilde{p}_{ij}^0$  is chosen such that it finishes on this edge exactly when player  $p_{ij}^0$  arrives. Hence,  $p_{ij}^0$  is not affected by these new players. We do the same construction for the players  $\tilde{p}_{ij}^1$ . But if the decider uses this edge, he has to wait  $\varepsilon$  time units there, and reaches the node on the path of the clause player at exactly the same time as the clause player. As the decider has the lowest priority he will thus be delayed. Hence, the players  $\tilde{p}_{ij}^0$  and  $\tilde{p}_{ij}^1$  ensure that the decider sticks to the edges in the two rows, which concludes the proof.

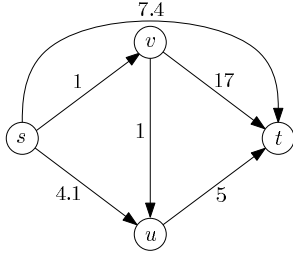
## B. PROOF OF THEOREM 3.4

The example for the first case is simple; it consists of three edges: there are three nodes  $s$ ,  $v$ , and  $t$  and two parallel edges from  $s$  to  $v$  (if multi edges are not allowed, they can be split up into two edges each by inserting intermediate nodes) and one edge from  $v$  to  $t$ . All edges have speed 1. Assume that there are two players with weights 2 and 3, and assume that the player with weight 3 has higher priority. If both players use the same edge from  $s$  to  $v$ , then the player with weight 2 has an incentive to switch to the free edge.





**Figure 4: Asymmetric temporal network congestion game without Nash equilibrium. Edge labels indicate the speeds  $a_e$ . For all unlabelled edges  $e$ , we have  $a_e = 1$ .**



**Figure 5: For two players with weights  $w_1 = 1$  and  $w_2 = 2$ , this temporal network congestion game with Shortest-First policy is not a potential game.**

If they use different edges, the player with weight 3 has an incentive to use the same edge as the other player.

Now let us turn to the second case. We consider the instance shown in Figure 4. In this game there are three unweighted players, and each player  $i$  has two possible strategies: the vertical three edges (denoted by  $A_i$ ) and another path (denoted by  $B_i$ ). The following sequence of moves constitutes a cycle in the best response dynamics:

$$\begin{aligned} (A_1, A_2, A_3) &\rightarrow (B_1, A_2, A_3) \rightarrow (B_1, B_2, A_3) \\ &\rightarrow (B_1, B_2, B_3) \rightarrow (A_1, B_2, B_3) \rightarrow (A_1, A_2, B_3) \\ &\rightarrow (A_1, A_2, A_3). \end{aligned}$$

It is easy to verify that the remaining configurations  $(A_1, B_2, A_3)$  and  $(B_1, A_2, B_3)$  are no Nash equilibria either.

### C. PROOF OF PROPOSITION 3.7

The game is depicted in Figure 5. For  $w_1 = 1$  and  $w_2 = 2$  the following is a cycle in the better-response dynamics:

$$\begin{aligned} ((s, u, t), (s, v, u, t)) &\rightarrow ((s, v, t), (s, v, u, t)) \rightarrow ((s, v, t), (s, t)) \\ &\rightarrow ((s, u, t), (s, t)) \rightarrow ((s, u, t), (s, v, u, t)). \end{aligned}$$

### D. PROOF OF THEOREM 3.10

The main observation here is that no task  $\pi(i)$  can influence the travel time of any task  $\pi(j)$  with  $j < i$ , because it will be preempted whenever it is scheduled simultaneously

with any of these tasks on an edge. This means that the vector  $(\ell_{\pi(1)}(P), \dots, \ell_{\pi(n)}(P))$  decreases strictly lexicographically whenever a player changes his path and decreases his individual latency. This proves that any such game is a potential game, which contrasts e.g. Proposition 3.7.

For efficient computation of a Nash equilibrium, we consider iterative entry dynamics according to the ranking with best-response computation of players. By the previous argument this process outputs a Nash equilibrium. For efficient computation of a best response strategy, we use the same variant of Dijkstra's algorithm that we have already used in the proof of Theorem 3.1. This time, however, a lower ranked task  $\pi(i)$  can arrive at a node before a higher ranked task  $j < i$  if it has a different source node. Then as soon as  $\pi(j)$  arrives,  $\pi(i)$  is preempted and blocked until it becomes the unfinished task of highest rank at the edge. Hence, the correctness of the algorithm is not affected by this. Finally, note that the previous algorithm does not rely on the fact that higher ranked players play a best response. The difference to Theorem 3.1 is that higher ranked player can never be delayed by lower ranked players even if they do not play best responses. Hence, the algorithm can be used in general to compute a best response for any player and any state.