

A Facility Location Problem under Competition

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May 19, 2011

Abstract. We consider a generalization of the discrete Voronoi game, in which consumers with positive purchase power who are located on the vertices of a network wish to connect to the nearest facility. Knowing this, competitive players locate their facilities on vertices, trying to capture the largest possible market share. We study conditions that guarantee the existence of a pure strategy Nash equilibrium in this finite non-cooperative game for progressively more complicated classes of networks, focusing in the two-player case although some results can be extended to a larger number of players. We find that equilibria in cycles exist when there is at least one dominant vertex with a sufficiently big demand. In the case of trees, equilibria always exist and consist of players selecting a centroid, defined as a solution to a centralized facility location game. For the case of a general graph, we construct a tree of maximal bi-connected components and apply the results derived for the simpler classes to get sufficient conditions for the existence of an equilibrium. This is shown to provide a complete and efficient characterization of equilibria in a broad class of structures, that includes cactus graphs. Finally, we provide sufficient conditions that guarantee that removing edges from the network increases consumer cost, which precludes situations like the Braess paradox, whereby removing an edge can *increase* the performance for all players. We show these conditions hold in a broad class of structures. These results imply that the networks with the worst possible equilibria are achieved in trees because they are minimal instances with respect to inclusion. While we show that equilibria can be arbitrary inefficient in general, we provide parametric upper bounds that depend on the topology of the network.

1 Introduction

Facility location problems study how to best locate facilities under the assumption that consumers will connect to the nearest open facility. Although most of the work in this area has been done from the perspective of centralized optimization, some of the existing literature has focused on the competitive version where different players choose a location each and compete for consumer demand. Namely, Hotelling introduced the first facility location model under competition [25]. This influential model assumes that two players have to each select a location in a linear segment, and that a continuum of uniformly-distributed consumers along the same segment select the closest facility.

Facility location games have several applications. Besides the

most immediate one, which is to use it to locate facilities, an interesting application that generalizes the work of Hotelling is a product differentiation model. Competing firms wish to maximize market share by producing the most profitable product, which is the one that attains a maximum demand. To do that, firms must select the properties of the product knowing that its competitors will do a similar thing. The analysis of this situation has been the subject of extensive research that focused on spatial competition in a continuous (usually linear) market in which consumers are distributed (in most cases, uniformly) over a low (mostly one) dimensional space [14, 20, 4, 32, 17]. Another famous variation is Salop's circular city model [37]. However, there have been relatively few attempts to study this competition problem when either the design characteristics are captured in a multi-dimensional space, or products are described as general objects. Examples of markets with these characteristics are given by radio and TV where the audience does not directly pay to the producer for consumption, some bus transit systems where bus routes are selected by operators but a regulatory authority sets fares, and in higher education in some countries where institutions adjust their program offerings but tuition rates are set by the government. We provide more details on this application below.

We assume that the characteristics of the market are encoded in a graph. Indeed, Wendell and McKelvey extended the basic model of Hotelling to a network context in which vertices represent consumers and edges succinctly encode the possible routes, but considered the set of possible locations to be the entire graph, including the interior of edges [38]. We only allow facilities to be located in the vertices and study the existence of equilibria for progressively more complicated network structures. Because each player selects a vertex in the network, this game is finite and therefore always admits an equilibrium in mixed strategies [31]. But because randomizing over the location of a facility is not natural (it is hard to imagine that a player can make an important decision with a big financial impact using a coin flip), we focus on pure-strategy equilibria. However, there is no guarantee that such an equilibrium exists since it may happen that for any configuration of locations, one player always has an incentive to select another location instead of the one currently selected.

Recognizing that existence cannot be guaranteed, research efforts in the literature focused on alternative solution concepts. Prescott and Visscher [35] considered a Stackelberg game in which the players sequentially choose a location for their facilities. Hakimi [22, 23] presented computational approach to find the Stackelberg equilibrium. Several authors continued the study of optimal strategies and they extended this formulation in differ-

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ent directions such as allowing a player to control several facilities [8, 9, 15, 16, 30, 34, 36]. Other variants of the model include using utility models [10, 28], distributing demand using a logit rule [13], applying a gravity rule [5, 6, 7, 9, 11], and defining a sphere of influence to describe facility attractiveness level [12].

Our work closely relates to those of Durr and Thang [3], and Mavronicolas et al. [29]. They consider a discrete version of the Voronoi game in which vertices have equal weights, for a general number of players. Durr and Thang studied the existence of equilibria, and characterized the ratio of the consumer costs between the worst and the best equilibria. While they showed that deciding the existence of a Nash equilibrium for a given general graph is \mathcal{NP} -hard, we provide existence conditions and characterization in complexity which is linear in the size of the graph for progressively more complicated classes of networks, up to a broad class of general graphs, and allow general positive weights for the vertices. Instead, Mavronicolas et al. studied cycle graphs; we generalize their results on the existence of equilibria and their efficiency to the case of general positive weights in cycles and to broader classes of graphs.

Our Results We start our study of the equilibria of the facility location games with cycles. We find that an equilibrium exists if and only if one of the vertices is dominant, that is, it has a sufficiently big demand so it is convenient to locate a facility there. In the case of trees, we show that an equilibrium is always guaranteed to exist and it consists of facilities located in one of the at most two centroids of the tree. (A centroid is a solution to the 1-median problem in which an agent wants to place a single facility in the network to minimize the distance between all vertices and that facility.) We combine the two previous cases to provide results for more general graphs. We show that if one reduces an arbitrary graph to a tree of maximal bi-connected components, the equilibrium must be located in the bi-connected component that corresponds to the centroid of that tree, and that it must also be an equilibrium in the projection of the full graph to that component. This provides sufficient condition for equilibria in general graphs, which in the case where the bi-connected center of the graph is either a single vertex or a cycle (e.g., in cactus graphs where each edge is contained in at most one cycle) renders a full characterization. We mainly focus on duopolies but some results extend to more firms. The small number of firms can be justified by barriers of entry due to high investment requirements and technological constraints.

At equilibrium, the consumer cost is not necessarily optimal because players select their location optimizing the demand they attract, disregarding the former objective. Koutsoupias and Papadimitriou introduced the concept of *price of anarchy* to quantify the gap between the consumer cost at equilibrium and the minimum possible consumer cost that would be attained if players were controlled by a social planner [27, 33]. We present instances that illustrate that equilibria of the facility location game can be extremely inefficient, even in trees or cycles. To understand what instances are particularly bad, we study in what cases this game is monotone with respect to a topology given by edge inclusion. We provide sufficient conditions under which removing an edge cannot improve equilibria, excluding situations like

the *Braess paradox* [2] whereby adding an edge can decrease the performance of all players. The implication is that for monotone instances, the worst possible inefficiencies are achieved in trees. Although the price of anarchy cannot be bounded in general, we provide parametric upper bounds that depend on the size of the network, its diameter, and the variability of demands in the instance. Notice that these parameters are given by the instance and do not depend on the equilibrium itself so they can be used to provide efficiency results for classes of instances that satisfy certain characteristics, which is useful when one works with realistic applications since they are not infinitely large and their demands are not unbounded.

A General Product Design Application A general version of the product design problem referred to earlier may be modeled using the facility location game described in this work. The vertices V of the network represent products that are considered for production (but will not be necessarily produced). The weight $w(v_i)$ of a vertex quantifies its demand, given by the proportion of consumers who prefer product v_i over all other products. Two products v_i and v_j are connected with an undirected edge $e_{ij} \in E$ whenever these products are partial substitutes in the market. (This could represent that their functionality is similar or that they are technologically close.) Firms choose a product they will produce by selecting a vertex $v_i \in V$. In summary, the graph $G(V, E)$ models a market where firms choose what to produce and consumers buy the available product that is closest to their preferred one. We assume that prices in the market are fixed (potentially zero) and that revenues that producers receive are just a function of their market share. The resulting equilibria represent production profiles chosen by firms. A consequence from our monotonicity result (described in Section 6) is that a market is more efficient when the graph that describes it is more dense (i.e., the products preferred by consumers are more similar to one another).

2 The Competitive Facility Location Game

We consider a finite undirected graph $G(V, E)$ whose vertices represent the locations of consumers. Each vertex $v \in V$ has an associated weight $w(v) > 0$ that represents the demand in that location. We define $W(S) := \sum_{v \in S} w(v)$, and refer to the total demand as $W := W(V)$. Edges in E form a connected graph. Without much loss of generality, we consider that all edges have length 1 (fractional values can be approximated with edge subdivisions). The game is played among players in $N := \{1, \dots, n\}$ who select a location for their facilities. We let $x_i \in V$ be the vertex selected by player $i \in N$, and $\bar{x} := \{x_i\}_{i \in N}$ be the outcome of the game.

Given a profile \bar{x} , each vertex v splits its demand among facilities $F(v, \bar{x}) := \arg \min_{i \in N} d(v, x_i)$ that are closest to it. When there is a tie, we assume that the demand $w(v)$ will be equally split between players in that set. Similarly, a player i will receive its demand from vertices in $V_i(\bar{x}) := \{v \in V | d(x_i, v) \leq d(x_j, v) \text{ for all } j \in N\}$. The *utility* of player $i \in N$ equals the total demand attained at x_i ; i.e., $u_i(\bar{x}) = \sum_{v \in V_i(\bar{x})} w(v) / |F(v, \bar{x})|$. Note that our assumption of splitting demand equally between

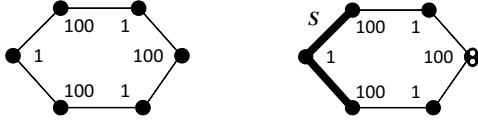


Figure 1: (Left) No equilibrium with two players in a cycle. (Right) The vertex selected by both players is not dominant because $W(S) = 201 > 303/2 = W/2$.

equidistant facilities is equivalent to assuming that the demand on each vertex is composed of small individuals that decide randomly which facility to use. Alternative approaches, such as assigning the whole demand to one arbitrary facility, do not change our main results, but may add technical difficulties in some cases.

We say that a profile \bar{x} is a (pure strategy) *Nash equilibrium* of the facility location game if $u_i(x_i, x_{-i}) \geq u_i(y, x_{-i})$ for any $y \in V$ and for any $i \in N$ [31]. In this work, we primarily focus on games with two players and study their pure-strategy equilibria. The characterization of equilibria with more players is much more involved and establishing conditions that guarantee existence is far from trivial. (And in fact, deciding whether or not an equilibrium exists is NP-hard [3].) Where possible, some directions of possible generalizations to more players will be presented. The following basic observation will be used to characterize equilibria.

Remark 1 *In an equilibrium of a facility location game with two players, both of them experience the same utility. This holds because if it were not the case, the player with the lower utility would prefer to select the location of the other player and consequently split the market. Hence, both utilities at equilibrium must equal $W/2$.*

The next three sections study the game for progressively more complicated classes of networks. For each class we present simple necessary and sufficient conditions for existence and uniqueness of equilibria, and provide an efficient characterization that allows us to list all possible equilibria. Note that we assumed demand to be strictly positive. Allowing $w(v) = 0$ for some vertex does not change any existence results, but it may generate multiplicity of equilibria.

3 The Case of Cycles

In this subsection, we study the equilibria of a facility location game when the graph is a cycle. The game with two players on a cycle may not possess an equilibrium: as illustrated in Figure 1 we can take a 6-cycle with weights that alternate between 1 and 100. Irrespective of the vertices selected by players, one of them can always find a profitable deviation. However, when a weight of 100 in an arbitrary vertex is replaced by one of 200, the profile where both players select that vertex is at equilibrium.

We observe that if two players are located on different vertices at equilibrium, not only each player receives half of the demand (Remark 1) but also it controls exactly half of the vertices (if there is an odd number of the vertices, one vertex is shared). Based on this observation, the connected half-cycles will allow us to characterize the equilibria of the game.

Definition 1 *In a cycle of cardinality k , a half-cycle S is a set of adjacent vertices that constitute a half of the cycle.*

When k is even, a half-cycle may be either a set of $k/2$ adjacent vertices, or a set of $k/2 + 1$ ones where the two extreme vertices get only half of their original weight (i.e. half-vertices). When k is odd, each half-cycle contains $(k - 1)/2$ adjacent vertices and one half-vertex. In both cases, there are a total of $2k$ half-cycles. When both players are located on different vertices, each of them controls a half-cycle. Therefore, if both players are located on the same vertex v , one of them would prefer a different vertex only if a half-cycle of total weight exceeding $W/2$ that excludes v exists. The approach we take to characterize equilibria with 2 players on cycles consists on finding *dominant* vertices, where a vertex v is dominant if an improving half-cycle does not exist when both players are located on v . It turns out that a dominant vertex exists if and only if an equilibrium exists.

Definition 2 *A vertex v is called dominant if $W(S) \leq W/2$ for every half-cycle S that does not contain v .*

To show the relation between dominant vertices and equilibria, let us revisit the instance without equilibria presented earlier. The picture on the right of Figure 1 shows that none of the vertices in this cycle is dominant because for every vertex v we can find a half-cycle that does not contain v and that has a weight larger than $W/2$. A location profile in which one of the players is located at a non-dominant vertex is not an equilibrium. Since the utilities of both players at equilibrium are equal, it suffices to show that the other player can reach a utility of more than $W/2$ to get a contradiction. Missing proofs can be found in the full version of the paper due to lack of space.

Theorem 1 *An equilibrium of a facility location game with two players on a cycle exists if and only if the cycle contains a dominant vertex. In addition, any profile that consists of dominant vertices is an equilibrium.*

The set of equilibria for two players in a cycle is derived directly from the set of dominant vertices. Whenever there are m dominant vertices, there are $m + m(m - 1)/2$ different equilibria, not including permutations between players, and m^2 equilibria including permutations. In the full version of the paper, we provide a method to enumerate all equilibria in a cycle of size k in $O(k)$ time, by finding all the dominant vertices.

Notice that when all weights are equal, all the vertices are dominant and Theorem 1 implies that an equilibrium always exists. In this case, any location profile is an equilibrium. This result relates to the characterization of equilibria on unit-weight cycles for general number of players by Mavronicolas et al. [29].

4 The Case of Trees

In this section, we focus on trees $T = T(V, E)$ and study the equilibria of the respective facility location games. A *centroid* of the tree is a natural candidate for an equilibrium location.

Definition 3 *A vertex v is a centroid of a tree T if it solves the 1-median problem; i.e., if $v \in \arg \min_{v_k \in V} \sum_{i \in V} d(v_k, v_i)w(v_i)$.*

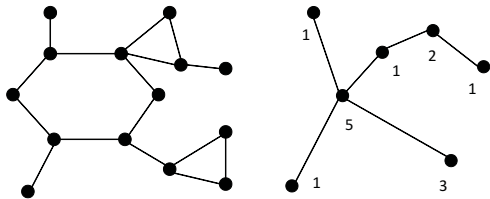


Figure 2: A graph (left) and its tree of bi-connected components (right)

Equivalently, a centroid can be defined as a vertex that minimizes the maximum weight of a connected component induced by removing this vertex. Kariv and Hakimi [26], who showed this equivalence, also showed that a vertex v is a centroid of a tree T of total weight W if and only if the removal of v induces components of weight at most $W/2$. It can be shown that a tree with positive weights has either one or two centroids. In the latter case, both centroids must be adjacent. Our next two results will be useful to characterize the equilibria of the facility location game in trees.

Proposition 2 *A tree T of total weight W has two centroids if and only if the removal of a vertex induces a component of weight exactly equal to $W/2$.*

Proposition 3 *For any vertex v that is not a centroid of T , there is a centroid such that removing it will leave v in a component T' of weight $W(T') < W/2$.*

We next prove that trees always admit an equilibrium.

Proposition 4 *In a facility location game with two players on a tree, an equilibrium always exists. Moreover, a location profile is an equilibrium if and only if both players selected a centroid of the tree (not necessarily on the same one).*

This provides a $O(k)$ -time algorithm for finding all the equilibria in a tree of size k because we can solve the 1-median problem to find all centroids of a tree within that time [19]. We remark that if the number of players is larger than two, an equilibrium may not exist, even if the graph is a tree. We will touch upon the case of more than two players in Section 7.

5 More General Graphs

In this section we switch our attention to more general topologies. We will combine the results obtained for cycles and trees by transforming a general graph to a tree and a cycle, while conserving the essential information of the instance. Using the results of the previous two sections, we provide a full equilibria characterization for a broad class of general graphs, and show that location profiles at equilibrium for arbitrary general graphs can be narrowed down to a typically small subset of vertices.

5.1 Bi-connected Components and Transformations Based on Them Our approach will be to represent a graph as a tree of *bi-connected components*, where a bi-connected component is a subgraph in which each pair of vertices is connected by at least two vertex-disjoint paths. This transformation represents a

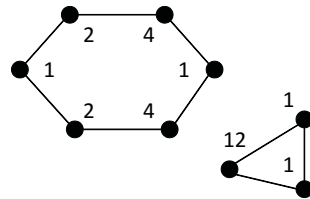


Figure 3: Two projections of B onto bi-connected components

general graph B as a weighted tree, while keeping some of the relevant information about the original graph.

Definition 4 [24] *A bi-connected component tree representation of graph $B(V, E)$ is a weighted tree $B'(V', E')$, in which every maximal bi-connected component in B is represented by a vertex in B' whose weight equals that of the component it represents.*

Notice that a vertex may belong to more than one maximal bi-connected component. When this happens this vertex is copied to the maximal bi-connected tree with its own weight. To prevent double-counting the adjacent components will not include its weight. Figure 2 provides an example that illustrates the transformation of a unit-weighted graph into its bi-connected component tree. The construction of a bi-connected component tree from a graph with k vertices can be done in $O(k)$ operations [1].

Any general graph B contains a finite set $\{Y_j\}$ of maximal bi-connected components. We would like to pay special attention to components associated to centroids of the bi-connected component tree. It turns out that these bi-connected components in B are natural locations to look for the equilibria of B .

Definition 5 *A bi-connected center of a general graph B is a bi-connected component Y_j associated to a centroid in the bi-connected component tree B' .*

Our next definition allows us to focus on the important part of the graph by projecting it to one of its maximal bi-connected components, typically the bi-connected center.

Definition 6 *Given a general graph B and a maximal bi-connected component $Y_j \subseteq B$, we define a weighted graph B_j as the projection of B onto Y_j if B_j is isomorphic to Y_j and the weights of its vertices equal the original weights plus the weights of the components not in Y_j connected through that vertex.*

To illustrate, Figure 3 depicts two projections of the graph B shown in Figure 2.

5.2 Characterization of Equilibria To characterize the equilibria of a facility location game on a general graph B with two players, our next result shows that it is enough to consider bi-connected centers.

Theorem 5 *A facility location game with two players on a general graph B admits an equilibrium if and only if a projection B_j of B onto a bi-connected center Y_j admits an equilibrium as well. Furthermore, if \bar{x} is an equilibrium of B_j , it is also an equilibrium of B .*

We next provide an example of a simple class of a general graph, for which we can characterize equilibria using Theorem 5.

Corollary 6 *If a bi-connected center of B is a single vertex v , then $\bar{x} = (v, v)$ is an equilibrium in B . Moreover, the equilibrium is unique if and only if v is a unique bi-connected center (i.e. associated with the unique centroid of B').*

We next define a class of graphs for which the combination of our results for cycles and trees provides a full characterization of equilibria. This class both generalizes cycles and trees.

Definition 7 [18] *A graph $B(V, E)$ is called a cactus if each edge in E belongs to at most one cycle.*

We note that when B is a cactus, the maximal bi-connected components are cycles, and so are the corresponding projections, which allows us to use the results of Section 3.

Corollary 7 *If a bi-connected center of B is a cycle (in particular, when B is a cactus), we have a complete characterization of the equilibria of the facility location game with two players.*

The idea of this result is to use Theorem 1 to find an equilibrium in the projection of B onto its bi-connected center, and then use Theorem 5 to complete the characterization. Corollary 7 is important because it applies to a fairly general class of graphs.

We complete this section with a short discussion on the complexity of finding an equilibrium. Naively, to find equilibrium profiles for n players in a k -vertex graph we need to check k^n possible profiles. For each we would need to calculate utilities, and compare with the utilities calculated for all the alternatives. Altogether, we need $O(k^{2n})$ to complete a list of existing equilibria. However, we have discussed the complexity of finding equilibria in trees and cycles in the previous sections, and showed that in both cases the such process can be done in $O(k)$ -time. The computation of the bi-connected component tree [1] and the projections on the bi-connected center can also be done in $O(k)$ time. Therefore, we can find an equilibrium of a graph with a bi-connected center given by a cycle (cacti in particular) in $O(k)$ -time.

6 Efficiency of Equilibria

In this section we examine the inefficiency of the equilibria of facility location games, usually referred to by price of anarchy. To do that, we introduce our measure of social cost.

Definition 8 *Let $d_v^{\bar{x}}$ denote the distance from vertex $v \in V$ to the nearest facility in \bar{x} . We define the total consumer cost corresponding to a location profile \bar{x} as $D_{\bar{x}} := \sum_{v \in V} d_v^{\bar{x}} w(v)$.*

With this definition in mind, the *consumer optimum* refers to the solution of the n -median problem, where n is the number of players in the game. This profile is the best outcome for consumers, but in practice it can only be achieved if players coordinate themselves and agree to just benefit consumers, as oppose to maximizing their own utilities. Let us denote the optimal consumer cost by D_{opt} .

The *Price of Anarchy* is defined as the worst-case inefficiency of an equilibrium among all instances [27]. It is a measure of how much it is lost by the lack of central coordination. To compute it, we evaluate the ratio of the cost of an arbitrary equilibrium to that of a consumer optimum, and maximize the ratio over all instances, as given by a graph G and a demand vector associated to the vertices of G . In summary, we evaluate

$$\text{POA} := \sup_{G, \{w(v)\}_{v \in V}} \frac{D_{\text{eq}}}{D_{\text{opt}}},$$

where $\text{NE}(G, \{w(v)\}_{v \in V})$ is the set of equilibria for the instance, and $D_{\text{eq}} := \sup_{\bar{x} \in \text{NE}(G, \{w(v)\}_{v \in V})} D_{\bar{x}}$ is the consumer cost of the worst equilibrium. We note that the price of anarchy is defined only when an equilibrium exists. Furthermore, we disregard instances where $D_{\text{opt}} = 0$ because otherwise the inefficiency is trivially unbounded. This can happen for instance in a game with two players on two vertices connected by an edge. A consumer optimum selects both vertices achieving a cost of zero while both players at equilibrium may select the same vertex.

6.1 Monotonicity In this section, we study if equilibria of facility location games are monotone with respect to edge removals. In effect, it would be intuitive that such a thing happens because removing edges constrain results so value would be lost. Nevertheless, it is well-known that removing edges in network routing games can make all players (and the social cost) better off. This apparently counterintuitive phenomenon has been called the *Braess paradox* [2]. We show that facility location games are well-behaved in this respect: equilibria get worse when an edge is removed. That motivates the following definition of *monotonicity* for instances. We study this property for the case of two players, noting that whether it holds or not depends on the number of players in the game.

Definition 9 *Let $B(V, E)$ be an arbitrary graph that admits an equilibrium. We say that B is monotone under edge removals if for any subset of edges $E' \subseteq E$ that induces a connected graph $\hat{B}(V, E')$, \hat{B} has an equilibrium such that $D_{\text{eq}}(\hat{B}) \geq D_{\text{eq}}(B)$.*

Whenever we remove an edge from a monotone graph, the worst case equilibrium can only get worse. Furthermore, the property is transitive: if $B(V, E)$ is monotone and $E' \subseteq E$ induces a connected graph that admits an equilibrium, $\hat{B}(V, E')$ is monotone too. We next show that monotonicity is obtained in a broad class of structures in which bi-connected centers are fairly simple. In particular, the result implies that cacti are monotone.

Theorem 8 *Let $B(V, E)$ be a general graph with bi-connected centers that are either cycles or vertices, then B is monotone.*

6.2 Bounding the Efficiency of Equilibria Using monotonicity, it follows that when looking for inefficient instances and price-of-anarchy results, it is enough to just consider trees. We note that this result does not use the fact that there are only two players in the game; it holds as long as the graph is monotone.

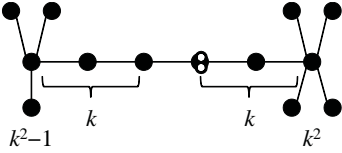


Figure 4: A Bad Tree

Theorem 9 For any monotone graph $G(V, E)$, for any edge $e \in E$ that is contained in a cycle, $G(V, E \setminus \{e\})$ has a higher price of anarchy.

The price of anarchy in trees, however, is not bounded, even for two players and unit weights. Figure 4 shows an example of two players on a tree with unit-weight vertices for which there is a unique equilibrium and a unique social optimum for any $k \geq 2$. Under the unique equilibrium, both facilities are located in the center (the only centroid). There are $O(k^2)$ vertices at a distance of $O(k)$ of the centroid, totalling a consumer cost of $D_{\text{eq}} = O(k^3)$. On the other hand, it is optimal to place facilities at the extremes, in the center of the clusters, where $O(k^2)$ vertices are at a unit distance from the facilities while $O(k)$ vertices along the segment connecting the two extremes are located at distance of $O(k)$. This results in $D_{\text{opt}} = O(k^2)$, from where we see that the price of anarchy is not bounded.

A similar example can be given for non-uniform weights when keeping the size of the graph constant. The price of anarchy grows when $\delta := \max_{v \in V} w(v) / \min_{v \in V} w(v) \rightarrow \infty$. As an example, we can take a path on $|V|$ vertices whose two leaves have a weight of $(W - (|V| - 2))/2$ and the interior vertices have unit weight. When $W \rightarrow \infty$ we have $D_{\text{opt}} \rightarrow 0$ while D_{eq} is bounded away from zero.

We next turn to show an upper bound on the price of anarchy for a given tree as function of the size of the tree, the spread of weights, and the diameter (i.e., the length of the longest path).

Proposition 10 For a tree of size $|V|$, diameter d and weights between 1 and δ , the price of anarchy is bounded by

$$1 + \frac{4\delta|V|d}{d^2 - 4}.$$

When applying Theorem 9, Proposition 10 gives an upper bound for the price of anarchy on any monotone graph. Theorem 8 points out this bound holds for any network for which the bi-connected center is either a cycle or a vertex, and in particular holds for cactus graphs. We note that in the case of vertices with uniform weight we have $\delta = 1$, and that in the case of lines we have $|V| = d$. As an example, for a unit-weight line with d vertices we get a bound of 5. However, this bound is loose; the full version of the paper shows that the price of anarchy in this case is exactly $9/4$.

It could happen that the price of anarchy is bounded for some classes of graphs that do not contain trees because worst-case instances would be excluded. Unfortunately, the price of anarchy is unbounded even for cycles. We do not include an example because it is an extension of the instance shown in Figure 4.

7 Final Remarks

We have provided an exhaustive characterization of equilibria for facility location games for different classes of topologies under duopolies. Below we comment on some generalizations that would be interesting to tackle in the future.

More Than Two Players. Another interesting direction is characterizing equilibria for an arbitrary number of players in trees and in cycles with arbitrary weights. Such results would allow us to generalize Theorem 5 to ultimately provide necessary and sufficient conditions for existence of equilibrium on more general graphs and allow us to extend Corollary 7 for cactus graphs to an arbitrary number of players.

We have seen that we can always find an equilibrium on a dominant vertex of a cycle when there are two players. The generalization to more players is not immediate. Namely, the characterization of Theorem 1 was based on how players split the vertices and the total weight and this property does not hold for more than two players.

For the case of trees with two players, we have seen that we can always find equilibria among the centroids. However, an equilibrium may not exist if more players participate. For example, a unit-weight tree with three players may not admit an equilibrium. Actually, one can show that there is an equilibrium only if any centroid of that tree has a degree strictly larger than two. We note that this characterization generalizes the fact that an equilibrium does not exist for three players on a unit-weight line. In the full version of the paper, we provide a complete characterization of equilibria on a tree with three players using a decomposition technique that may be useful for further generalizations. While a generalization to an arbitrary number of players on a tree is difficult, such generalization may be successful for simpler structures. For instance, in sharp contrast to the case of a line with three players, Gur shows that equilibria always exist on a line with k unit-weight vertices with 4 or more players [21]. He also provides the equilibrium profiles, which depend on the values of n and k .

General Bi-connected Graphs. Corollary 7 provides a characterization of equilibria for two players in a cactus, and more generally whenever the bi-connected center of the graph is either a cycle or a vertex. However, a bi-connected graph can have a more complicated structure than a cycle, making equilibria harder to find. A characterization of equilibria for two players in a general bi-connected graph would yield, in the form of Corollary 7, an equilibrium characterization for two players on an arbitrary general graph.

Monotonicity. It would be interesting to establish monotonicity for other classes of graphs. Theorem 9 would imply that the minimal elements of that class with respect to inclusion are worst-case instances. The upper bound provided by Proposition 10 would hold too.

Multiple Facilities for each Player. Generalizing our results to the case where each player controls M facilities is far from trivial. Taking the case of trees as an example, Proposition 4 cannot be generalized: even for $M = 2$, the M -median solution does not necessarily represent an equilibrium.

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