Foundations of Information Aggregation Mechanisms
Preliminary and Incomplete: a Revision will be Sent Later

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Abstract

A recent innovation in market design is the use of the market mechanism to aggregate private information to forecaset future events. In this paper I consider conditions for information aggregation in parimutuel pricing systems.

1 Introduction

A fundamental fact in our lives is dispersion of relevant information among players and their incentives to play with their private information. Thus efficient aggregation of these private information has been one of the key tasks in market and institution design. For example, Hayek (1945) was able to predict that market based mechanisms would outperform central planned command mechanisms at 1945 based on this insight.

Recent developments of information technology such as the Internet and the Peer-to-Peer systems have drastically decreased the cost of complex communications. How will it affect information aggregation ? Is it possible to design novel innovative systems ?

In this paper I propose a study of *Information Aggregation Mechanisms* (Plott and Sunder (1988), Plott (2000)) which use price mechanisms to aggregate information among players to predict future events. There have been already several implementations: (1) Iowa Electronic Market[1] has created a market to trade contingent claims based on economic and political events. (2) Idea futures and policy analysis market (Charles,Hanson, Ledyard, Ishikida (2003)) has used market for contingent claims for the prediction of political events. (3) Sales forecasts (Plott and Chen (2002)). Hewlett-Packard uses a market for contingent claims for sales data for predictions. The price has been a more accurate predictor than the official forecast. (4) Economic Derivatives[2]. Goldman Sachs and Deutsche Bank created markets for derivatives whose payoff depends on the economic statistics such as nonfarm payrolls.

Early analysis of these mechanism use a rational expectations equilibrium approach. The application of a rational expectations equilibrium in an economy with differential information is problematic since players do not have incentives to acquire information given the fully revealing price (e.g. Milgrom (1981)). As a result, it is meaningful to apply a game theoretic approach to explicitly study the price formation process and the incorporation of information into prices.

In this paper I consider a sealed-bid pari-mutuel pricing system, which is used in, for example, the auctions for Economic Derivatives. In a sealed-bid pari-mutuel system, each player chooses among possible realizations of the underlying state variable. The state price (or implied probabilities) is defined as the amount of bids for this realization of the state variable divided by the total amount of bids. If a player

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[1] http: //www.biz.uiowa.edu/iem/

[2] http: // www.gs.com/econderivs/

is able to guess the correct realization of the state, then the player receives the amount which equals to his/her bid divided by the state price. Otherwise the player just loses the amount of the bid.

I define information aggregation in this system to be convergence of the state prices to the probability distribution of the underlying state variable as the number of bidders goes to infinity. The basic idea is that if there is a divergence between the state prices and the distribution, there are opportunities for arbitrage. If a player is certain that the state price for some realization is strictly less than the underlying probability, then the player can make a strictly positive expected payoff from bidding for this realization. This strictly positive payoff will give a violation to the zero expected equilibrium profit condition derived from the zero-sum nature of the pari-mutuel system. In this argument, it is not necessary that the market designer knows about these underlying distribution of the state. In this sense, the market designer can aggregate information from players while respecting incentive constraints without any payments.

There is a similarity between the argument for information aggregation in auctions (Wilson (1977), Milgrom (1979, 81), Pesendorfer and Swinkels (1997), and Jackson and Kremer (2004)) and the argument in this pari-mutuel system. In both cases, the failure of information aggregation will lead to an arbitrage opportunity. In auctions, the basic idea is that, if there are positive probability events where the true value of the object and the winning price are different, then a bidder can make a strictly positive profit in this event. In pari-mutuel, a difference between the state prices and underlying probabilities creates a profitable deviation.

A difference is that the pari-mutuel system requires stronger conditions for information aggregation[3]. In order to understand this point, first let me recall the argument for information aggregation in first price auctions.

Building on an insight on Wilson (1977), Milgrom (1979) considered the distinguishability condition as a necessary and sufficient condition for information aggregation in first price auctions. Intuitively, the distinguishability condition implies that, for each value, there exists a positive probability event that a bidder is very sure that the true value is equal or higher than this value. In auctions, a bidder, knowing that there is a difference between the value and the winning price at this value and this event, can decrease the bid to take advantage of the difference. On the other hand, this condition is necessary: without this condition, a bidder needs to concern the possibility that the bidder is going to win the object with a higher price than the true value of the object.

A difference arises in the case where the realized state is "higher" than the state the bidder puts a bid on. In auctions, if the realized value is higher, then some other bidders must be bidding close to this value, so that this bidder is not going to win the object, and the bidder expected payoff is going to be zero. Thus a bidder in auctions does not need to worry the case where the realized state is higher than the bid. This is a reason that the bidder only need to distinguish whether the value of the object is lower than the bid or not.

In contrast, in pari-mutuels, the bidder still has to pay the bid in this case when the realized value is higher than the bid. Thus in pari-mutuel, the bidder needs to avoid this possibility. This implies that the bidder wishes to distinguish whether the realized signal is higher than the bid or not, in addition to whether the realized signal is lower than the bid or not. This implies that information aggregation is harder in pari-mutuels than in auctions[4].

The rest of the paper is organized as follows. In section 2, I present a formulation of pari-mutuel systems. In section 3, I consider conditions for information aggregation. Section 4 concludes.

[3] I have been working on about finding a weaker condition for information aggregation, or the necessity of distinguishability condition in this paper.

[4] But at the same time, it should be noted that the system is budget-balanced: the market designer does not need any costs to aggregate information. Pesendorfer and Swinkels (1996) introduced a weaker double largness condition for information aggregation in uniform price auctions. Jackson and Kremer (2004) showed that information aggregation does not hold in discriminatory price auctions with double largeness conditions since a bidder wants to shade the bid even in the limit. An appplication of these insights to pari-mutuel systems is an interesting open question.

2 A Model

This section provides a model for a sealed-bid pari-mutuel system. I start with the description of underlying state, signals, then the pari-mutuel game, strategy and payoffs, and finally move to an equilibrium.

Let $(\Omega,\mathcal{F})$ be a measurable space.

First consider uncertainty about the state of the nature. Uncertainty is represented by a random variable V on $(\Omega,\mathcal{F})$ to a finite set of $\{v_1, ..., v_M\}$. Let π be the density function of V. This implies $\sum_{m=1,...,M} \pi(v_m) = 1$. Assume $\pi(v_m) > 0$ for every m.

Next define the structure of the signal. Each player $i = 1, ..., n$ receives a signal X_i. I assume that signals are distributed iid given the value of V. Let f_m be the conditional density of X_1 given the realization of v_m. I assume for each v_m and for each possible value of the signal x, $f_m(x_1) > 0$.

The pari-mutuel game is defined as follows. First, each player i receives a realization of signal x_i conditional on the realization of V. Then player i chooses $b_i \in \{v_1, ..., v_M\}$ and bids 1 unit of good, which can be considered as money. Suppose bidder i bids for v_l. Then the state price (or the implied probability) $p_{l,n}$ is defined by $p_{l,n} = \#\{i : b_i = v_l\}/n$. If the realized state is v_l, player i receives $1/p_{l,n}$. Otherwise, player i does not receive anything. Thus the payoff of player i is $\frac{1_{V=v_l}}{p_{l,n}} - 1$.Each player is risk neutral.

Let me pause for a moment to make an intuitive comparison with first price auctions. In first price auctions, the payoff formula is $(V - b_i)1_{b_i \geq \max_{j \neq i} b_j}$ where V is the value of the object and b_i is the bid by bidder i. Information aggregation implies a bidder earns zero expected payoffs: V converges to the winning bid W_n (in probability). In my pari-mutuel model, the expected payoff formula is $\pi_l/p_{l,n} - 1$. A zero profit implies that the state price $p_{l.n}$ converges to the underlying probability π_l. This is, of course, a familiar result in asset pricing theory (assuming risk-neutrality).

Then I consider a strategy, a payoff, and an equilibrium. A pure strategy of player i is a map β_i from the set of possible signal to $\{v_1, ..., v_M\}$. In order to compute the expected payoff at state s_i given the action $\beta_i(s_i)$ and a strategy of other players β_{-i}, first let me compute the ex post payoff given the realization of (v, s_{-i}) . In this case, the other players's bids are $\beta_{-i}(s_{-i})$, so the state price for the state $\beta_i(s_i)$ is $p_{\beta_i(s_i),n} = \#\{j : \beta_j(s_j) = \beta_i(s_i)\}/n$. Thus the payoff is

$$u_i(s_i, \beta_i(s_i), s_{-i}, \beta_{-i}(s_{-i}), v) = \frac{1_{V=v_{\beta_i(s_i)}}}{p_{\beta_i(s_i),n}} - 1.$$

By taking expectations, the interim expected payoff is

$$U_i(s_i, \beta_i(s_i), \beta_{-i}) = \int \frac{1_{V=v_{\beta_i(s_i)}}}{p_{\beta_i(s_i),n}} dF(s_{-i}, v|s_i) - 1.$$

The equilibrium is defined by a strategy profile $\{\beta_i\}_{i=1,...,n}$ such that, for each i, for almost every s_i, for every $b_i, U_i(s_i, \beta_i(s_i), \beta_{-i}) \geq U_i(s_i, b_i, \beta_{-i})$.

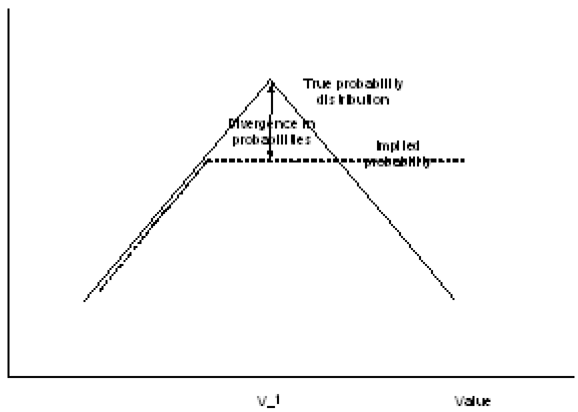
Finally I define that information aggregation takes place at an equilibrium if, given an equilibrium strategy profile β, the state price converges to the underlying distribution of the state variable in distribution, $p_{m,n} \rightarrow \pi_m$ as $n \rightarrow \infty$.

In contrast to auction theory where the convergence is defined by a convergence in probability of the underlying value and the winning price, I defined the convergence in terms of its distribution. It is because in pari-mutuel, I am interested in the convergence between the state prices and the underlying distribution of the state variable, not in the distance between the true value and the winning price.

3 Information Aggregation

This section explains the argument for information aggregation.

A basic idea is that the divergence between the state price and the underlying distribution will provide an arbitrage opportunity. Consider a following picture:



In this picture, at v_1, the state price p_1 is lower than prior probability π_1. Then, bidding at v_1 will provide a strictly positive payoff because $\pi_1/p_1 - 1 > 0$.

As a simple numerical example, consider a case of a coin toss. In this case, the prior probabilities of head and tail are 0.5 respectively. If the state price for head is 0.7 and for tail is 0.3, then bidding on tail is going to produce a excess payoff of 2/3. Thus the bid for tail will increase and for head will decrease. This arbitrage will eventually lead to coincidence between the prior probability and the implied probability and there will be no arbitrage in the limit.

In order to deal with incomplete information, I consider a version of distinguishability

Assumption. *The signal structure satisfies the distinguishability condition if for each possible value v_m there exists some bidder i and a positive probability event A_i such that $P(v_m|x_i \in A_i) = 1$.*

This distinguishability is stronger than the corresponding condition in Milgrom (1979): the signal structure must distinguish not only between $\{V = v_k\}$ and $\{V < v_k\}$, but also $\{V = v_k\}$ and $\{V > v_k\}$.

Proposition. *With the distinguishability condition, a pari-mutuel system aggregates information.*

Proof. The proof consists of two parts. First I check that the expected payoff in an equilibrium is zero. In the second part, I show that if the information aggregation fails, then there exists an deviation with strictly positive expected payoffs.

First, for each realization of (v, s) and a strategy profile $\{\beta_i\}$, since the game is zero-sum, $\sum_{i=1,\dots,n} u_i(s_i, \beta_i(s_i), \beta_{-i}(s_{-i}), v) = 0$.By integrating over (v, s), I get $\sum_{i=1,\dots,n} U_i(s_i, \beta_i(s_i), \beta_{-i}) = 0$.In addition, since each player can have nonnegative payoffs in an equilibrium, $U_i(s_i, \beta_i(s_i), \beta_{-i}) \geq 0$ for each i, s_i,and an equilibrium strategy $\{\beta_n\}$. Thus in an equilibrium, $U_i(s_i, \beta_i(x_i), \beta_{-i}) = 0$ for each i, s_i.

Suppose information aggregation fails. Then there exists some underlying value v_m such that $\limsup_n P_1(p_{m,n} - \delta\pi_m \leq 0|v = v_m) > \alpha$ for some $\delta < 1$. Then, by distinguishability, there exists some player i and some positive probability event A_i such that $P(\omega : v(\omega) = v_m|x_i \in A_i) = 1$. Consider a bid of v_m if $x_i \in A_i$. Then there exists some $n < \infty$ such that $P_1(p_{m,n} - \delta\pi_m \leq 0|v = v_m) > \alpha$.

Then i's expected payoff from bidding on v_m at $A_i$is

$$\int \frac{1}{p_{m,n}} 1_{v=v_m, \text{ and } p_{m,n}-\delta\pi_m \leq 0} dF(v, x_{-i}|x_i \in A_i) + \int \frac{1}{p_{m,n}} 1_{v=v_m, \text{ and } p_{m,n}-\delta\pi_m > 0} dF(v, x_{-i}|x_i \in A_i) - 1$$

Now bound from the first term using the relation that $p_{m,n} \leq \delta\pi_m$, and the second term using the relation of $p_{m,n} \leq 1$, I get the lower bound of the above expression by

$$\frac{1}{\delta\pi_m} \int 1_{v=v_m, \text{ and } p_{m,n}-\delta\pi_m \leq 0} dF(v, x_{-i}|x_i \in A_i) + \int 1_{v=v_m, \text{ and } p_{m,n}-\delta\pi_m > 0} dF(v, x_{-i}|x_i \in A_i) - 1$$

Since $\int 1_{v=v_m, \text{ and } p_{m,n}-\delta\pi_m \leq 0} dF(v, x_{-i}|x_i \in A_i) > \alpha$ and $\int 1_{v=v_m, \text{ and } p_{m,n}-\delta\pi_m > 0} dF(v, x_{-i}|x_i \in A_i) \leq (1 - \alpha)$. Thus the expression is bounded below by a positive constant $\frac{1}{\delta\pi_m}\alpha + (1 - \alpha) - 1 > 0$. ■

4 Conclusion

This paper proposes a research agenda of understanding the strategic foundation of information aggregation mechanism.

For example, static mechanisms versus dynamic mechanisms[5]. This paper studied a one-shot process. What will be the properties of dynamic processes ? Will it lead to herding behavior ? How do anomalies such as long-shot bias take place ? Will the comparison between one-shot mechanisms and dynamic mechanisms in auction theory carry over to pari-mutuels ?

Another issue will be revenue properties. There is no revenue for the market maker in this model. What is the optimal policy of the market maker to maximize the expected revenue with a finite number of bidders ?

It will be interesting to conduct a performance comparison between pari-mutuel systems and other mechanisms such as (double) auctions for contingent claims.

Finally, more generally, how does the decentralized communication network structure of the Internet, exemplified by P2P systems, affect evolution of consensus[6] ? Under what conditions these communications will lead to efficient information aggregation/herding ?

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