Efficient Graph Topologies in Network Routing Games*

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Abstract

In this work we explore the topological structure of networks that guarantee that any routing of selfish users is efficient, i.e., any Nash equilibrium achieves the social optimum.

We distinguish between two classes of atomic network routing games. In both classes the cost of the social optimum is the maximum cost over the players. In the first, network congestion games, *the player's cost is the sum of the latency costs over the edges in its route, while in the second,* bottleneck routing games, *the player's cost is the maximum edge cost over the edges in its route.*

Our interesting results are for the symmetric case of a single source and a single destination (single-commodity). We show that for network congestion games the efficient topologies are exactly Extension Parallel Graphs, while for bottleneck routing games the efficient topologies are exactly Series Parallel Graphs. For the asymmetric case of multiple sources or destinations (multi-commodity), we show that the efficient topologies are very limited and include either trees or trees with parallel edges.

1 Introduction

A very natural setting of routing includes multiple players that each would like to establish a connection between a source and a destination. Each of the players is selfish, and would like to route its connection as to minimize its cost. An equilibrium is a collection of routes (one per player) where no player can improve its cost by changing its route (unilaterally). This general setting is a *network routing game*.

Network routing games have been the subject of intensive study initially in game theory and recently in computational game theory, where the main aim has been to bound the inefficiency of a Nash equilibrium in comparison to the social optimum solution. The Price of Anarchy, which is the ratio between the worst Nash equilibrium and the social optimum cost, has been the major measure by which this inefficiency has been quantified [12, 13, 1, 3, 14, 15, 2].

In this work we take a different approach. Rather than quantifying the inefficiency of Nash equilibrium, we characterize the topologies that guarantee that the cost of any Nash equilibrium is the social optimum cost. We concentrate on topological properties of graphs, and say that a graph is *efficient* if, for any assignment of non-decreasing cost functions on the edges, the resulting congestion game has the property that the cost of any Nash equilibrium coincides with the social optimum cost.

One can view this separation between the graph topology and the costs, as a separation between the underlying infrastructure and the costs the players observe to purchase routes. While one expects the infrastructure to be stable over long period of times, the costs the players observe can be easily modified over short time periods. Topological characterizations for single-commodity network games (i.e., where all players share the same source and destination nodes) have been recently provided for various equilibrium properties, including (Nash and strong) equilibrium existence [10, 4, 6, 7], equilibrium uniqueness [8] and equilibrium efficiency [12, 9].

Our main results are for two classes of atomic network routing games. In both classes the social cost is the *maximum* cost of the individual players (unlike the more "standard" social cost which is the sum, or equivalently average, of the players' costs). This social cost has been first studied in [13], where it was shown that in non-atomic single-commodity network congestion games, the price of anarchy is $n-1$. We distinguish between two classes of atomic network routing games with the *maximum* social cost. In the first class, network congestion games, the player's cost is an *aggregative cost*, i.e., the sum of the edge costs over the edges in its route. In the second class, bottleneck routing games, the player's cost is a *maximum cost*, i.e., the maximum edge cost over the edges in its route. The latter case has been first studied by [2], who provided bounds for the price of anarchy in the splittable and unsplittable flow mod-

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els.

Our social cost function resembles the Max-Min Fairness criterion, where the goal of the network operator is to maximize the fairness between the different network users. The aggregative player's cost is applicable in cases such as delay, where the total user's delay is composed of the delay on the various links it traverses. The maximum cost can represent a bandwidth allocation problem, where the user's bandwidth is limited by the most loaded link on its path.

Our interesting results are for the symmetric case of a single source and a single destination. We show that for network congestion games the efficient topologies are exactly Extension Parallel Graphs (EPG), while for bottleneck routing games the efficient topologies are exactly Series Parallel Graphs (SPG). Our proofs show an even stronger property in such topologies. They show that each player, for any joint action of the other players (a selection of paths), has a *best response* whose cost is at most the social optimum cost (recall that the social cost is the maximum over users' costs).

For the case of multiple destinations (or equivalently multiple sources) we show that the efficient topologies are very limited. For network congestion games the only efficient topologies are either trees or two nodes with parallel edges. For bottleneck routing games we show that the only efficient topologies are trees with parallel edges.

2 Model

A game $\Lambda = (N, \{\Sigma_i\}_{i\in N}, \{c_i\}_{i\in N})$, has a set N of $n \geq 2$ players, and for each player $i \in N$, a finite set of actions Σ_i and a cost function c_i mapping from $\Sigma = \Sigma_1 \times \ldots \times \Sigma_n$ to the reals. Let $S = (S_1, \ldots, S_n) \in \Sigma$ denote the joint action taken by the players, and let $S_{-i} = (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n)$ denote the joint action taken by players other than player i. We also denote $S = (S_i, S_{-i})$.

The network routing games that we will be considering will have an underlying graph $G = (V, E)$, where V is the set of nodes and $E \subset V \times V$ the set of edges. Each player $i \in N$ has a source node $s_i \in V$ and a destination node $t_i \in V$. The action set Σ_i of player i is a set of paths in G connecting s_i to t_i.

Each edge $e \in E$ is associated with a non-decreasing cost function $\ell_e : \{1, 2, \ldots, n\} \to \Re$. Given a joint action $S = (S_1, \ldots, S_n)$, we denote by $n_e(S) = |\{i|e \in S_i\}|$ the number of players that route using edge e in the joint action S. In a *network congestion game* the cost of a player is the *aggregate cost*, i..e., $c_i(S) = \sum_{e\in S_i} \ell_e(n_e(S))$, and in a *bottleneck routing game* the cost of a player is the *maximum cost*, i.e., $c_i(S) = \max_{e\in S_i} \ell_e(n_e(S))$.

A network routing game on a graph G is *symmetric* if all players have the same source node and destination node (also called single-source single-destination or single-commodity). Otherwise, it is called *asymmetric* (either multiple sources, multiple destinations or both).

Given a game Λ, we need to define its *social cost*. Abstractly, there is a function $cost_\Lambda$ such that the social cost of $S \in \Sigma$ is $cost_\Lambda(S)$, and the optimal social cost is $OPT(\Lambda) = \min_{S\in\Sigma} cost_\Lambda(S)$. In this paper we concentrate on the case that $cost_\Lambda(S) = \max_{i\in N} c_i(S)$

Pure Nash Equilibrium: A joint action $S \in \Sigma$ is a *pure Nash Equilibrium* if no player $i \in N$ can benefit from unilaterally deviating from his action to another action, i.e., $\forall i \in N\ \forall S'_i \in \Sigma_i\ :\ c_i(S_{-i}, S'_i) \geq c_i(S)$.

Since every congestion game possesses at least one pure Nash equilibrium [11], it is guaranteed that there is a pure Nash equilibrium. The proof for the existence of a pure Nash equilibrium for bottleneck routing games is based on showing that any best response dynamics converges to a pure Nash Equilibrium, and is similar in spirit to that of job scheduling [5].

Proposition 2.1 *Every network congestion game and every bottleneck routing game, possesses at least one pure Nash equilibrium.*

To fully characterize the set of network topologies in which pure Nash equilibrium is the social optimum in this family of games, we first provide a definition of an optimum-inducing network topology.

Definition 2.2 *A graph topology $G = (V, E)$ is* optimum-inducing *for a family of network routing games $\mathcal{F}$ if for every network routing game $\Lambda \in \mathcal{F}$ on the graph G, $max_{S\in\Phi(\Lambda)} cost_\Lambda(S) = OPT(\Lambda)$, where $\Phi(\Lambda)$ is the set of pure Nash equilibria of the game Λ.*

We also need the following definition of embedding. A graph G' is *embedded* in a graph G if G' can be obtained from G by a sequence of removal of edges and contraction of edges (identification of the two vertices connected by an edge). Note that intuitively, removal of an edge is equivalent to assigning the edge the latency function $\ell_e(x) = +\infty$ and contraction of edge is equivalent to assigning the edge the latency function $\ell_e(x) = 0$. It is easy to see that if the embedded graph G' is not optimum inducing for a given network routing game, then any graph G, such that G' is embedded in G, is also not optimum inducing (this is shown by simulating the game of G' on the graph G using appropriate latency functions).

2.1 Extension Parallel and Series Parallel Graphs

Our graphs would have a *source* node and a *sink* node. We first define the following actions for composition of graphs.

- Identification: The *identification* operation allows to collapse two nodes to one. More formally, given graph $G = (V, E)$ we define the *identification* of a node $v_1 \in V$ and $v_2 \in V$ forming a new node $v \in V$ as creating a new graph $G' = (V', E')$, where $V' = V \setminus \{v_1, v_2\} \cup \{v\}$ and E' includes the edges of E where the edges of v_1 and v_2 are now connected to v.

- Parallel composition: Given two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with sources $s_1 \in V_1$ and $s_2 \in V_2$ and sinks $t_1 \in V_1$ and $t_2 \in V_2$, respectively, we define a new graph $G = G_1||G_2$ as follows. Let $G' = (V_1 \cup V_2, E_1 \cup E_2)$ be the union graph. To create $G = G_1||G_2$ we identify the sources s_1 and s_2, forming a new source node s, and identify the sinks t_1 and t_2, forming a new sink t.

- Series composition: Given two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with sources $s_1 \in V_1$ and $s_2 \in V_2$ and sinks $t_1 \in V_1$ and $t_2 \in V_2$, respectively, we define a new graph $G = G_1 \rightarrow G_2$ as follows. Let $G' = (V_1 \cup V_2, E_1 \cup E_2)$ be the union graph. To create $G = G_1 \rightarrow G_2$ we identify the vertices t_1 and s_2, forming a new vertex u. The graph G has a source $s = s_1$ and a sink $t = t_2$.

- Extension composition : A series composition when one of the graphs, G_1 or G_2, is composed of a single edge is an extension composition, and we denote it by $G = G_1 \rightarrow_e G_2$.

An *extension parallel graph (EPG)* is a graph G consisting of either: (1) a single edge (s, t), (2) a graph $G = G_1||G_2$, where G_1 and G_2 are extension parallel graphs, or (3) a graph $G = G_1 \rightarrow_e G_2$, where either G_1 or G_2 is composed of a single edge and the other is an extension parallel graph. A *series parallel graph (SPG)* is a graph G consisting of either: (1) a single edge (s, t), (2) a graph $G = G_1||G_2$ or (3) a graph $G = G_1 \rightarrow G_2$, where G_1 and G_2 are series parallel graphs.

3 Efficient Topologies in Network Congestion Games

In a non-increasing network congestion game, the delay function $\ell_e(x)$ is non-decreasing in x, and $c_i(S) = \sum_{e \in S_i} \ell_e(n_e(S))$. There are two common social costs that are studied in the context of network congestion games, namely the maximum latency [13] and the total latency [12, 15, 14]. Here, we consider the maximum latency social cost, $cost(S) = \max_i c_i(S)$. We note that for the total latency social cost function, the price of anarchy is unbounded even for a simple topology of two parallel edges [12]. In this section we characterize optimum-inducing network topologies in symmetric network congestion games with a maximum latency social cost. Milchtaich [9] provided a similar characterization for the non-atomic case[1]. Our main focus is on symmetric network congestion games, since we show that for asymmetric network congestion games any optimum-inducing graph is a forest or a graph with two vertices.

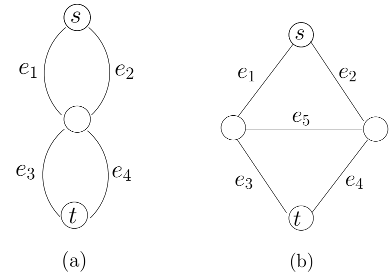

Figure 1. Graph (a) is embedded in every graph which is not EPG, and graph (b) is embedded in every graph which is not SPG.

Consider a symmetric network congestion game on an extension parallel graph. Then, in any joint action of the players the cost of the best response strategy of any player is at most the optimal social cost.

Lemma 3.1 *Let Λ be a non-increasing symmetric network congestion game on an extension parallel graph G with source s and sink t. Consider any joint action $S \in \Sigma$. Let P_i be a best response of any player i. Then $c_i(P_i, S_{-i}) \leq OPT(\Lambda)$.*

Proof: We prove the lemma by induction on the network size $|V|$. Let Λ be a non-increasing network congestion game on an EPG $G = (V, E)$. The claim obviously holds if $|V| = 2$.

Extension composition: Suppose the graph $G = G_1 \rightarrow_e G_2$ is an extension composition of the graph G_1 consisting of a single edge $e = (s_1, t_1)$ and an EPG $G_2 = (V_2, E_2)$ with terminals s_2, t_2, such that $s = s_1$ and $t = t_2$ (the case that G_2 is a single edge is similar). Let Λ' be the original game on the graph G_2. The joint action S' of the game Λ' is obtained from S by removing the edge e from the strategy S_j of any player j. Let P'_i be a best response of player i in the game Λ' and let $P_i = P'_i \cup \{e\}$. Then, by the induction hypothesis $c_i(P'_i, S'_{-i}) \leq OPT(\Lambda')$. Hence, $c_i(P_i, S_{-i}) \leq OPT(\Lambda)$.

Parallel composition: Suppose the graph $G = G_1||G_2$ is a parallel composition of two EPG graphs G_1 and G_2. Let S^* be the optimal joint action, i.e., $OPT(\Lambda) = cost_\Lambda(S^*)$.

[1]Milchtaich's definition of Pareto efficiency coincides with the social optimum under the maximum latency social cost.

Let T_j be the set of players using paths in G_j according to S and $x_j = |T_j|$. Let T_j^* be the set of players using paths in G_j according to S^* and $x_j^* = |T_j^*|$. There are two cases:
Case 1: $x_1 = x_1^*$ and $x_2 = x_2^*$. Let Λ_1 and Λ_2 be the original game on the respective graphs G_1 and G_2 with players T_1 and T_2 respectively. Let S be a joint action of the players in the game Λ and let S' and S'' be the induced joint actions of the players in the games Λ_1 and Λ_2 respectively. By the induction hypothesis for every player i in the game Λ_1 with best response strategy P_i, $c_i(P_i, S'_{-i}) \leq OPT(\Lambda_1)$ and for every player i in the game Λ_2 with best response strategy P_i, $c_i(P_i, S''_{-i}) \leq OPT(\Lambda_2)$. Since $OPT(\Lambda_j) \leq OPT(\Lambda)$, we obtain that for every player i, $c_i(P_i, S_{-i}) \leq OPT(\Lambda)$.
Case 2: There exists a network G_j for which $x_j^* > x_j$. W.l.o.g., suppose $x_1^* > x_1$. Consider player i. Let $x_1' = |T_1 \cup \{i\}|$. Then, $x_1^* \geq x_1 + 1 \geq x_1'$. Let Λ_1 be the original game on the graph G_1 with players $T_1 \cup \{i\}$. Let S be the joint action of the players in the original game Λ, let S' be the induced joint action of the players T_1 in the game Λ_1 and let P_i be the best response strategy (path) of player i in Λ_1 for the joint action S'. It follows from the induction hypothesis that in Λ_1 we have $c_i(P_i, S') \leq OPT(\Lambda_1)$. Since $x_1^* \geq x_1'$, we have that $OPT(\Lambda_1) \leq OPT(\Lambda)$, and therefore it follows that $c_i(P_i, S_{-i}) \leq OPT(\Lambda)$. ■

The following theorem follows directly from Lemma 3.1.

Theorem 3.2 *If a graph G is an EPG, then it is optimum-inducing for symmetric network congestion games.*

Next we show that the only efficient topology for symmetric network congestion games is an EPG.

Theorem 3.3 *Let G be a graph that is optimum-inducing for symmetric network congestion games. Then, G is an EPG.*

Proof: Let G be a graph that is not an EPG. We use the following lemma, which is implicit in [9].

Lemma 3.4 *[9] Let G be a graph that is not an EPG. Then, the network in figure 1(a) is embedded in G.*

Consider the graph given in Figure 1(a) with the following delay functions: $\ell_{e_1}(x) = 2, \ell_{e_2}(x) = x, \ell_{e_3}(x) = x, \ell_{e_4}(x) = 2$. Consider a symmetric network congestion game with two players played on the graph G with source s and sink t. One can verify that this game admits a pure Nash equilibrium in which $S_1 = S_2 = \{e_2, e_3\}$, resulting in $cost(S) = 4$. Consider the joint action S' in which $S_1' = \{e_1, e_3\}$ and $S_2' = \{e_2, e_4\}$. It holds that $cost(S') = 3 < 4 = cost(S)$. Therefore, G is not optimum-inducing. ■

Theorem 3.2 and Theorem 3.3 yield the following corollary.

Corollary 3.5 *In a symmetric network congestion game with the maximum latency social cost, a graph topology G is optimum-inducing if and only if G is an EPG.*

Finally we characterize efficient topologies for asymmetric network congestion games.

Lemma 3.6 *Any connected graph with at least 3 vertices containing a cycle is not optimum-inducing for asymmetric network congestion games.*

Proof: We prove the claim by showing that every graph containing a cycle of length 2 or a cycle of length 3 is not optimum-inducing for asymmetric network congestion games. Suppose the graph G contains a cycle of length 2. Consider the graph G given in Figure 2(a) with the following delay functions: $\ell_{e_1}(x) = x, \ell_{e_2}(x) = 2, \ell_{e_3}(x) = 2$. Consider an asymmetric network congestion game with two players played on the graph G. The two players share a common source s and the sinks of players 1 and 2 are t_1 and t_2 respectively. One can verify that this game admits a pure Nash equilibrium in which $S_1 = \{e_1\}$ and $S_2 = \{e_1, e_3\}$, resulting in $cost(S) = 4$. However, the joint action S' in which $S_1' = \{e_2\}$ and $S_2' = \{e_1, e_3\}$ yields $cost(S') = 3 < 4 = cost(S)$. Therefore, G is not optimum-inducing. Now suppose the graph G contains a cycle of length 3. Consider the graph G given in Figure 2(b) with the following delay functions: $\ell_{e_1}(x) = 2x, \ell_{e_2}(x) = 2x, \ell_{e_3}(x) = x$. Consider an asymmetric network congestion game with two players played on the graph G. The two players share a common source s and the sinks of players 1 and 2 are t_1 and t_2 respectively. One can verify that this game admits a pure Nash equilibrium in which $S_1 = \{e_2, e_3\}, S_2 = \{e_1, e_3\}$, resulting in $cost(S) = 4$. However, the joint action S' in which $S_1' = \{e_1\}$ and $S_2' = \{e_2\}$ yields $cost(S') = 2 < 4 = cost(S)$. Therefore, G is not optimum-inducing. ■

The following theorem follows directly from Lemma 3.6.

Theorem 3.7 *For asymmetric network congestion games, every optimum-inducing connected graph is a tree or a graph with two vertices.*

4 Efficient Topologies in Bottleneck routing games

In a bottleneck routing game, the delay function $\ell_e(x)$ is non-decreasing in x, $c_i(S) = \max_{e \in S_i} \ell_e(n_e(S))$, and the social cost is $cost(S) = \max_i c_i(S)$. We note that for the total bottleneck social cost function, i.e., $cost(S) = \sum_i c_i(S)$, the price of anarchy is unbounded even for a simple topology of two parallel edges. This is shown by

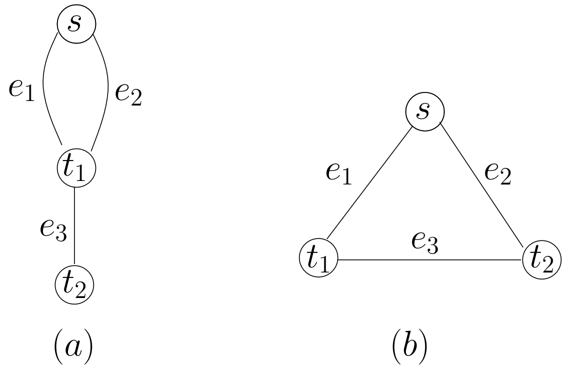

Figure 2. Asymmetric network routing games.

the same example given in [12] for the unbounded price of anarchy in network congestion games with the total latency social cost. In this section we characterize optimum-inducing network topologies in bottleneck routing games with the maximum bottleneck social cost. We consider only symmetric bottleneck routing games, since we show that for asymmetric bottleneck routing games every optimum-inducing connected graph is a tree with possibly multiple parallel edges.

Consider a symmetric bottleneck routing game on a series parallel graph. Then, in any joint action of the players the cost of the best response strategy of any player is at most the optimal social cost.

Lemma 4.1 *Let Λ be a symmetric bottleneck routing game on a series parallel graph G with source s and sink t. Consider any joint action $S \in \Sigma$. Let P_i be a best response of any player i. Then $c_i(P_i, S_{-i}) \leq OPT(\Lambda)$.*

Proof: We prove the lemma by induction on the network size $|V|$. The claim obviously holds if $|V| = 2$. We show the claim for a series composition, i.e., $G = G_1 \rightarrow G_2$, and for a parallel composition, i.e., $G = G_1 || G_2$, where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are SPG's with sources s_1, s_2, and sinks t_1, t_2, respectively. Let Λ be a bottleneck routing game on an SPG $G = (V, E)$.

Series composition: Let $G = G_1 \rightarrow G_2$. Let Λ_1 and Λ_2 be the original game on the respective graphs G_1 and G_2. Let S be a joint action of the game Λ and let S' and S'' be the induced joint actions of the players in the games Λ_1 and Λ_2 respectively. Consider player i such that $c_i(S) = \max_j c_j(S)$. Let P'_i and P''_i be the best response strategies (paths) of player i in the games Λ_1 and Λ_2 respectively and let $P_i = P'_i \cup P''_i$ be a strategy of player i in the original game Λ. By the induction hypothesis, $c_i(P'_i, S'_{-i}) \leq OPT(\Lambda_1)$ and $c_i(P''_i, S''_{-i}) \leq OPT(\Lambda_2)$. Since $OPT(\Lambda) = \max(OPT(\Lambda_1), OPT(\Lambda_2))$, we get $c_i(P_i, S_{-i}) \leq OPT(\Lambda)$.

Parallel composition: the proof is similar to the parallel composition case in Lemma 3.1. ∎

The following theorem follows directly from Lemma 4.1.

Theorem 4.2 *Any SPG is optimum-inducing for symmetric bottleneck routing games.*

Next we show that the only efficient topology for symmetric bottleneck routing games is an SPG.

Theorem 4.3 *Let G be a graph that is optimum-inducing for bottleneck routing games. Then, G is an SPG.*

Proof: Let G be a graph that is not an SPG. We use the following lemma, which is implicit in [9]:

Lemma 4.4 *[9] Let G be a graph that is not an SPG. Then, the graph in figure 1(b) is embedded in G.*

Consider the graph given in Figure 1(b) with the following delay functions: $\ell_{e_1}(x) = \ell_{e_3}(x) = x, \ell_{e_2}(x) = \ell_{e_4}(x) = \ell_{e_5}(x) = 2x$. Consider a symmetric bottleneck routing game with six players played on the graph G with source s and sink t. One can verify that this game admits a pure Nash equilibrium in which $S_1 = S_2 = S_3 = \{e_2, e_5, e_3\}$ and $S_4 = S_5 = S_6 = \{e_1, e_3\}$ resulting in $cost(S) = 6$. However, the joint action S' in which $S'_1 = S'_2 = \{e_2, e_4\}$ and $S'_3 = S'_4 = S'_5 = S'_6 = \{e_1, e_3\}$ yields $cost(S') = 4 < 6 = cost(S)$. Therefore, G is not optimum-inducing. ∎

Theorem 4.2 and Theorem 4.3 yield the following corollary.

Corollary 4.5 *For symmetric bottleneck routing games, a graph topology G is optimum-inducing if and only if G is an SPG.*

Finally we characterize efficient topologies for asymmetric bottleneck routing games.

Lemma 4.6 *Any graph containing a cycle of length 3 is not optimum-inducing for asymmetric bottleneck routing games.*

Proof: Consider the graph G given in Figure 2(b) with the following delay functions: $\ell_{e_1}(x) = \ell_{e_2}(x) = \ell_{e_3}(x) = x$. Consider an asymmetric bottleneck routing game with two players played on the graph G. The two players share a common source s and the sinks of players 1 and 2 are t_1 and t_2 respectively. One can verify that this game admits a pure Nash equilibrium in which $S_1 = \{e_2, e_3\}, S_2 = \{e_1, e_3\}$, resulting in $cost(S) = 2$. However, the joint action S' with $S'_1 = \{e_1\}$ and $S'_2 = \{e_2\}$ yields $cost(S') = 1 < 2 = cost(S)$. Therefore, G is not optimum-inducing. ∎

The following theorem follows from Lemma 4.6 and the simple fact that in every asymmetric bottleneck routing game on a tree with possibly multiple parallel edges and for every joint action of the players, the cost of the best response strategy of every player is at most the optimal social cost.

Theorem 4.7 *For asymmetric bottleneck routing games every optimum-inducing connected graph is a tree with possibly multiple parallel edges.*

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